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Some common fixed point results for three total asymptotically pseudocontractive mappings

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Abstract

In this article, we introduce a mixed-type implicit iterative scheme to approximate the common fixed points of finite families of three uniformly L-Lipschitzian total asymptotically pseudocontractive mappings in Banach spaces. Also, we prove some strong convergence results of the proposed iterative scheme. Our results which are new, improve and generalize the results of many prominent authors exiting in the literature.

Keywords Implicit iterative scheme · Total asymptotically pseudocontractive mappings · Banach space · Strong convergence

Mathematics Subject Classification 47H05 · 47H09 · 39B82

1 Introduction

Let $\mathcal{W}$ denote the nonempty closed subset of a Banach space $\mathcal{Q}$ with dual $\mathcal{Q}^*$. Let J stand for the normalized duality mapping from $\mathcal{W}$ into $2^{\mathcal{W}^*}$ and it is defined by

$$J(p) = \{t^* \in \mathcal{W}^* : \langle p, t^* \rangle = \|p\|^2 = \|t^*\|^2\}, \ \forall p \in \mathcal{W}, \tag{1.1}$$

where $\langle ., . \rangle$ denote the generalized duality pairing. Throughout this manuscript, we denote the single-valued-normalized duality by j, set of all positive integers is denoted by $\mathbb{R}^+$, set of all natural number is denoted by $\mathbb{N}$ and set of all fixed points of a mapping $M : \mathcal{W} \to \mathcal{W}$ is denoted by $F(M) = \{p \in \mathcal{W} : Mp = p\}$. The fixed-point theory is important to many applied and theoretical fields, such as linear and variational inequality, nonlinear analysis, approximation theory, dynamic system theory,

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Extended author information available on the last page of the article mathematical modelling, mathematics of fractals, mathematical economics (equilibrium problems, game theory and optimization problems), differential and integral equations. For recent results on applications of fixed point, the reader may refer to [2–5, 10–12, 19–23].

Definition 1.1 A self-mapping M defined on $\mathcal{W}$ is called uniformly Lipschitzian, if for all $p, q \in \mathcal{W}$, there exists a constant $L \geq 0$ such that

$$\|M^v p - M^v q\| \leq L\|p - q\|, \quad \forall v \in \mathbb{N}. \tag{1.2}$$

Definition 1.2 A self-mapping M defined on $\mathcal{W}$ is called pseudocontractive if for any $p, q \in \mathcal{W}$, there exists $j(p - q) \in J(p - q)$ such that

$$\langle Mp - Mq, j(p - q)\rangle \leq \|p - q\|^2; \tag{1.3}$$

Definition 1.3 [30, 31] A self-mapping M defined on $\mathcal{W}$ is called asymptotically pseudocontractive (AP) if there exists a sequence $\{h_v\} \subset [1, \infty)$ with $h_v \to 1$ as $v \to \infty$ such that

$$\langle M^v p - M^v q, j(p - q)\rangle \leq h_v\|p - q\|^2, \; \forall v \geq 1, \text{ and } p, q \in \mathcal{W}. \tag{1.4}$$

Definition 1.4 [27] A self-mapping M defined on $\mathcal{W}$ is called asymptotically pseudocontractive in the intermediate sense (APIS) if there exists a sequence $h_v \subset [1, \infty)$ with $h_v \to 1$ as $v \to \infty$ and $j(p - q) \in J(p - q)$ such that

$$\limsup_{v \to \infty} \sup_{(p,q) \in \mathcal{W}} \left(\langle M^v p - M^v q, j(p - q)\rangle - h_v\|p - q\|^2\right) \leq 0. \tag{1.5}$$

Set

$$\tau_v = \max\left\{0, \sup_{p,q \in \mathcal{W}} \left(\langle M^v p - M^v q, j(p - q)\rangle - h_v\|p - q\|^2\right)\right\}.$$

It implies that $\tau_v \geq 0$, $\tau_v \to 0$ as $v \to \infty$. So, (1.5) becomes

$$\langle M^v p - M^v q, j(p - q)\rangle \leq h_v\|p - q\|^2 + \tau_v, \; \forall\, v \geq 1, \; p, q \in \mathcal{W}. \tag{1.6}$$

Definition 1.5 [28] A self-mapping M defined on $\mathcal{W}$ is called total asymptotically pseudocontractive (TAP) if there exists sequences $\{\mu_v\} \subset [0, \infty)$ and $\{\xi_v\} \subset [0, \infty)$ with $\mu_v \to 0$ and $\xi_v \to 0$ as $v \to \infty$ such that

$$\langle M^{v}p - M^{v}q, j(p-q)\rangle \leq \|p-q\|^{2} + \mu_{v}\phi(\|p-q\|) + \xi_{v}, \tag{1.7}$$

for all $v \geq 1$ and $p, q \in \mathcal{W}$, where $\phi : [0, \infty) \to [0, \infty)$ is a continuous and strictly increasing function with $\phi(0) = 0$.

Remark 1.6 Suppose $\phi(p) = p^{2}$, then (1.7) reduces to the class of APIS mappings as follows:

$$\langle M^{v}p - M^{v}q, j(p-q)\rangle \leq (1+\mu_{v})\|p-q\|^{2} + \xi_{v} \tag{1.8}$$

for all $v \geq 1,\ p, q \in \mathcal{W}$. Set

$$\tau_{v} = \max\left\{0,\ \sup_{p,q\in\mathcal{W}}\left(\langle M^{v}p - M^{v}q, j(p-q)\rangle - (1+\mu_{v})\|p-q\|^{2}\right)\right\}.$$

It is not hard to see that the class of APIS mappings is a proper subclass of the class of TAP mappings.

Remark 1.7 If $\tau_{v} = 0$, for all $v \geq 1$, then the class of APIS mappings reduces to the class of AP mappings.

Following the above implications in Remarks 1.6 and 1.7, it follows that the class of TAP mappings properly includes all other classes of mappings mentioned above.

In recent years, several iterative methods for approximating fixed points of TAP mappings have been investigated by several researchers (see [1, 7, 8, 13] and the references therein).

Implicit iterative schemes have been known to be more efficient than the corresponding explicit iterative schemes. One of the first implicit iterative schemes was studied in 2001 by Xu and Ori [37]. After this, many implicit iteration processes for approximating fixed point of nonlinear mappings have been introduced and studied by many authors (see, e.g., [1, 13, 16–18, 24–26, 32, 36–38]).

In 2007, Thahur [33] proposed the following composite implicit iteration process for a finite family of asymptotically nonexpansive mappings as follows:

$$\begin{cases} p_{0} \in \mathcal{W}, \\ p_{v} = (1-m_{v})p_{v-1} + m_{v}M_{i(v)}^{k(v)}w_{v}, \quad \forall v \geq 1, \\ w_{v} = (1-t_{v})p_{v-1} + t_{v}M_{i(v)}^{k(v)}p_{v}, \end{cases} \tag{1.9}$$

where $\{m_{v}\}$ and $\{t_{v}\}$ are sequences in [0,1] and$v = (k-1)N + i$, $i = i(v) \in \{1, 2, ..., N\}$, $k = k(k) \geq 1$ is some positive integers and $k(v) \to \infty$ as $v \to \infty$.

Motivated and inspired by the above results, in this article, we introduce the following mixed-type iterative method for three finite families of three uniformly L-Lipschitzian TAP mappings:

$$\begin{cases} p_0 \in \mathcal{W}, \\ p_v = (1-m_v)p_{v-1} + m_v M_{i(v)}^{k(v)} w_v, \\ w_v = (1-t_v)p_{v-1} + t_v H_{i(v)}^{k(v)} z_v, \\ z_v = (1-c_v)p_v + c_v G_{i(v)}^{k(v)} p_v, \end{cases} \quad v \in \mathbb{N} \tag{1.10}$$

where $\{m_v\}, \{t_v\}$ and $\{c_v\}$ are real sequences in [0,1] and $v = (k-1)N + i$, $i = v(i) \in I = \{1, 2, ..., N\}$, $k = k(v) \geq 1$ is some positive integers and $k(v) \to \infty$ as $v \to \infty$.

Remark 1.8 Clearly, our new iterative scheme (1.10) properly contains the implicit Mann iterative scheme [14], implicit Ishikawa iterative scheme [9], implicit Noor iterative scheme [15], the iterative schemes (1.9) and several others iterative schemes in the literature.

We will prove strong convergence theorems for our new iterative algorithm (1.10) for common fixed points of finite families of three uniformly L-Lipschitzian TAP mappings in Banach spaces. We also provide an example to validate the assumptions in our main results. The results in this article extend, generalize and improve the corresponding results in [13, 25, 26, 29, 33] and several others in the literature.

2 Preliminaries

We recall some relevant definition and lemmas that will be used in this work.

Definition 2.1 [6] A family $\{M_i\}_{i=1}^N : C \to C$ with $\Im = \cap_{i=1}^N F(M_i) \neq \emptyset$ is said to satisfy *condition* (B) on $\mathcal{W}$ if there exists a nondecreasing self function f defined on $[0, \infty)$ with $f(0) = 0$, $f(s) > 0$ for all $s \in (0, \infty)$ such that for each $p \in \mathcal{W}$

$$\max_{1 \leq i \leq N} \{\|p - M_i p\|\} \geq f(d(p, \Im)). \tag{2.1}$$

Lemma 2.2 [36] *Let* $J : \mathcal{Q} \to 2^{\mathcal{Q}^*}$ *be the normalized duality mapping. Then for any* $p, q \in \mathcal{Q}$, *we have*

$$\|p + q\|^2 \leq \|p\|^2 + 2\langle q, j(p+q)\rangle, \quad \forall j(p+q) \in J(p+q). \tag{2.2}$$

Lemma 2.3 [35] *Let* $\{\rho_v\}$ *and* $\{\eta_v\}$, $\{v_v\}$ *be sequences of nonnegative real numbers satisfying the following inequality*:

$$\rho_v \leq (1 + \eta_v)\rho_v + v_v, \quad v \geq 1. \tag{2.3}$$

If $\sum_{v=1}^{\infty} \eta_v < \infty$ and $\sum_{v=1}^{\infty} v_v < \infty$ then $\lim_{v \to \infty} \rho_v$ exists. Additionally, if $\{\rho_v\}$ posses a subsequence $\{\rho_{v_i}\}$ such that $\rho_{v_i} \to 0$, then $\lim_{v \to \infty} \rho_v = 0$.

3 Main results

Firstly, we show that (1.10) can be applied to estimate the fixed points of TAP mappings which is assumed to be continuous. Let M_i be a L^i_m–Lipschitz TAP mapping with sequences $a^i_v \in [0,\infty)$ and $b^i_v \in [0,\infty)$ with $a^i_v \to 0$ and $b^i_v \to 0$ as $v \to \infty$ for all $i \in [1,N]$, where $N \in \mathbb{N}$. Let H_i be a L^i_h–Lipschitz TAP mapping with sequences $f^i_v \in [0,\infty)$ and $d^i_v \in [0,\infty)$ with $f^i_v \to 0$ and $d^i_v \to 0$ as $v \to \infty$ for all $i \in [1,N]$, where $N \in \mathbb{N}$. Let G_i be a L^i_g–Lipschitz TAP mapping with sequences $\eta^i_v \in [0,\infty)$ and $l^i_v \in [0,\infty)$ with $\eta^i_v \to 0$ and $l^i_v \to 0$ as $v \to \infty$ for all $i \in [1,N]$, where $N \in \mathbb{N}$.

Let the mapping $T_v : \mathcal{W} \to \mathcal{W}$ be defined by

$$\begin{aligned} T_v(p) = (1-m_n)p_{v-1} + m_v M^{k(v)}_{i(v)}\{(1-t_v)p_{v-1} \\ + t_v H^{k(v)}_{i(v)}[(1-c_v)p + c_v G^{k(v)}_{i(v)}p]\}, \quad \forall v \geq 1. \end{aligned} \tag{3.1}$$

From (3.1), we have

$$\begin{aligned} \|T_v(p) - T_v(q)\| &= m_v \|M^{k(v)}_{i(v)}\{(1-t_v)p_{v-1} + t_v H^{k(v)}_{i(v)}[(1-c_v)p + c_v G^{k(v)}_{i(v)}p]\} \\ &\quad - M^{k(v)}_{i(v)}\{(1-t_v)p_{v-1} + t_v H^{k(v)}_{i(v)}[(1-c_v)q + c_v G^{k(v)}_{i(v)}q]\}\| \\ &\leq m_v t_v L \|H^{k(v)}_{i(v)}[(1-c_v)p + c_v G^{k(v)}_{i(v)}p] - H^{k(v)}_{i(v)}[(1-c_v)q + c_v G^{k(v)}_{i(v)}q]\| \\ &\leq m_n t_n L^2[(1-c_v)\|p-q\| + c_v\|G^{k(v)}_{i(v)}p - G^{k(v)}_{i(v)}q\|] \\ &\leq m_v t_n L^2[(1-c_v)\|p-q\| + c_v L\|p-q\|] \\ &= m_v t_v L^2[(1 + c_v(L-1))]\|p-q\|, \end{aligned} \tag{3.2}$$

for all $p, q \in \mathcal{W}$, where $L = \max\{L^1_p, ..., L^N_p, L^1_h, ..., L^N_h, L^1_g, ..., L^N_g\}$.

Assume $m_v t_v L^2[(1 + c_v(L-1))] < 1$ for all $v \geq 1$, by (3.2), it implies that the mapping T_v is a contraction. Recalling Banach contraction principle, it follow that a unique point $p_v \in \mathcal{W}$ exist such that

$$\begin{aligned} p_v = T_v(p_v) = (1-m_v)p_{v-1} + m_v M^{k(v)}_{i(v)}\{(1-t_v)p_{v-1} \\ + t_v H^{k(v)}_{i(v)}[(1-c_v)p + c_v G^{k(v)}_{i(v)}p]\}, \quad \forall v \geq 1. \end{aligned}$$

This shows that the implicit iteration process (1.10) is well defined. Thus, the sequence (1.10) can be applied to estimate the common fixed points of three finite family of uniformly L–Lipschitzian TAP mappings.

Lemma 3.1 *Let $\mathcal{Q}$ be a Banach space and $\mathcal{W}$ be a nonempty close convex subset of $\mathcal{Q}$. Let $i \in I = [1,N]$, where $N \in \mathbb{N}$. Let $M_i : \mathcal{W} \to \mathcal{W}$ be a finite family of uniformly L^i_m–Lipschitzian TAP mappings with sequences $\{a^i_v\} \subset [0,\infty)$ and $\{b^i_v\} \subset [0,\infty)$, where $a^i_v \to 0$ and $b^i_v \to 0$ as $v \to \infty$, $H_i : \mathcal{W} \to \mathcal{W}$ be a finite family of uniformly L^i_h–Lipschitzian TAP mappings with sequences $\{f^i_v\} \subset [0,\infty)$ and $\{d^i_v\} \subset [0,\infty)$, where $f^i_v \to 0$ and $d^i_v \to 0$ as $v \to \infty$ and $G_i : \mathcal{W} \to \mathcal{W}$ be a finite family of* *uniformly* L_g^i*–Lipscitzian TAP mappings with sequences* $\{\eta_v^i\} \subset [0,\infty)$ *and* $\{l_v^i\} \subset [0,\infty)$, *where* $\eta_v^i \to 0$ *and* $l_v^i \to 0$ *as* $v \to \infty$, *for each* $i \in I$. *Let* $\mu_v = \max\{a_v, f_v, \eta_v\}$, *where* $a_v = \max\{a_v^i : i \in I\}$, $f_v = \max\{f_v^i : i \in I\}$ *and* $\eta_v = \max\{\eta_v^i : i \in I\}$. *Let* $\xi_v = \max\{b_v, d_v, l_v\}$, *where* $b_v = \max\{\lambda_v^i : i \in I\}$, $d_v = \max\{d_v^i : i \in I\} and l_v = \max\{l_v^i : i \in I\}$. *Suppose* $\Im = (\bigcap_{i=1}^N F(M_i)) \bigcap (\bigcap_{i=1}^N F(H_i)) \bigcap (\bigcap_{i=1}^N F(G_i)) \neq \emptyset$. *Let* $\phi(r) = \max\{\phi_i(r) : i \in I\}$, *for each* $r \geq 0$. *Assume that there exist* $K, K^* > 0$ *such that* $\phi(e) \leq K^* e^2$ *for all* $e \geq K$. *Let* $\{m_v\}$, $\{t_v\}$ *and* $\{c_v\}$ *be sequences in* [0,1]. *If the following assumptions are satisfied*:

(i) $\sum_{v=1}^{\infty} m_v = \infty$;

(ii) $\sum_{v=1}^{\infty} m_v^2 < \infty$;

(iii) $\sum_{v=1}^{\infty} m_v \mu_v < \infty$, $\sum_{v=1}^{\infty} m_v \xi_v < \infty$;

(iv) $\sum_{v=1}^{\infty} m_v t_v < \infty$;

(v) $m_v t_v L^2 [1 + c_v(L-1)] < 1, \forall v \geq 1$, where $L = \max\{L_m^1, \ldots, L_m^N, L_h^1, \ldots, L_h^N, L_g^1, \ldots, L_g^N\}$.

Let $\{p_v\}$ be a sequence defined by (1.10). Then, $\lim_{v\to\infty} \|p_v - q^\star\|$ exists for all $q^\star \in \Im$.

Proof Let $q^\star \in \Im$. From (1.10), we have

$$\begin{aligned}
\|z_v - q^\star\| &= \|(1-c_v)p_v + c_v G_{i(v)}^{k(v)} p_v - q^\star\| \\
&= \|(1-c_v)(p_v - q^\star) + c_v(G_{i(v)}^{k(v)} t_v - q^\star)\| \\
&\leq (1-c_v)\|p_v - q^\star\| + c_v\|G_{i(v)}^{k(v)} p_v - q^\star\| \\
&\leq \|p_v - q^\star\| + c_v\|G_{i(v)}^{k(v)} p_v - q^\star\| \\
&\leq \|p_v - q^\star\| + c_v L\|p_v - q^\star\| \\
&= (1 + c_v L)\|p_v - q^\star\| \\
&\leq (1 + L)\|p_v - q^\star\|. \qquad (3.3)
\end{aligned}$$

Using (1.10) and (3.3), we obtain

$$\begin{aligned}\|w_v - q^\star\| =& \|(1-t_v)p_{v-1} + t_v H_{i(v)}^{k(v)} z_v - q^\star\| \\ &\le (1-t_v)\|p_{v-1} - q^\star\| + t_v\|H_{i(v)}^{k(v)} z_v - q^\star\| \\ &\le \|p_{v-1} - q^\star\| + t_v L\|z_v - q^\star\| \\ &\le \|p_{v-1} - q^\star\| + t_v L(1+L)\|p_v - q^\star\|.\end{aligned} \tag{3.4}$$

Now, from (1.10) and by Lemma 2.2, we have

$$\begin{aligned}\|p_v - q^\star\|^2 =& \|(1-m_v)p_{v-1} + m_v M_{i(v)}^{k(v)} w_v - q^\star\|^2 \\ =& \|(1-m_v)(p_{v-1} - q^\star) + m_v(M_{i(v)}^{k(v)} w_v - q^\star)\|^2 \\ \le& (1-m_v)^2\|p_{v-1} - q^\star\|^2 + 2m_v\langle M_{i(v)}^{k(v)} w_v - q^\star, j(p_v - q^\star)\rangle \\ =& (1-m_v)^2\|p_{v-1} - q^\star\|^2 + 2m_v\langle M_{i(v)}^{k(v)} w_v \\ &- M_{i(v)}^{k(v)} p_v + M_{i(v)}^{k(v)} p_v - q^\star, j(p_v - q^\star)\rangle \\ =& (1-m_v)^2\|p_{v-1} - q^\star\|^2 + 2m_v\langle M_{i(v)}^{k(v)} w_v - M_{i(v)}^{k(v)} p_v, j(p_v - q^\star)\rangle \\ &+ 2\alpha_n\langle M_{i(v)}^{k(v)} p_v - q^\star, j(p_v - q^\star)\rangle \\ \le& (1-m_v)^2\|p_{v-1} - q^\star\|^2 + 2m_v\|M_{i(v)}^{k(v)} w_v - M_{i(v)}^{k(v)} p_v\|\|p_v - q^\star\| \\ &+ 2m_v\langle M_{i(v)}^{k(v)} p_v - q^\star, j(p_v - q^\star)\rangle \\ \le& (1-m_v)^2\|p_{v-1} - q^\star\|^2 + 2m_v L\|w_v - p_v\|\|p_v - q^\star\| \\ &+ 2m_v\langle M_{i(v)}^{k(v)} p_v - q^\star, j(p_v - q^\star)\rangle.\end{aligned} \tag{3.5}$$

By (1.10), we have that

$$\begin{aligned}\|w_v - p_v\| \le& \|w_v - p_{v-1}\| + \|p_{v-1} - p_v\| \\ =& \|(1-m_v)p_{v-1} + m_v H_{i(v)}^{k(v)} z_v - p_{v-1}\| \\ &+ \|p_{n-1} - [(1-m_v)p_{v-1} + m_v M_{i(v)}^{k(v)} w_v]\| \\ \le& t_v\|H_{i(v)}^{k(v)} z_v - p_{v-1}\| + m_v\|p_{v-1} - M_{i(v)}^{k(v)} w_v\| \\ \le& t_v\|H_{i(v)}^{k(v)} z_v - q^\star\| + t_v\|p_{v-1} - q^\star\| \\ &+ m_v\|p_{v-1} - q^\star\| + m_v\|M_{i(v)}^{k(v)} w_v - q^\star\| \\ \le& t_v L(1+L)\|p_v - q^\star\| + t_v\|p_{v-1} - q^\star\| \\ &+ m_v\|p_{v-1} - q^\star\| + m_v L\{\|t_{v-1} - q\| + t_v L(1+L)\|p_v - q\|\} \\ \le& t_v L(1+L)\|p_v - q^\star\| + t_v\|p_{v-1} - q^\star\| \\ &+ m_n\|p_{v-1} - q\| + m_v L\|t_{v-1} - q\| + m_v p_v L^2(1+L)\|p_v - q^\star\| \\ =& [m_v + t_v]\|p_{v-1} - q^\star\| + [t_v L(1+L) + m_v t_v L^2(1+L)]\|p_v - q^\star\| \\ =& [m_v(1+L) + t_v]\|p_{v-1} - q^\star\| + [t_v L(1+L)(1+m_v L)]\|p_v - q^\star\| \\ \le& [m_v(1+L) + t_v]\|p_{v-1} - q^\star\| + [t_v L(1+L)(1+L)]\|p_v - q^\star\| \\ =& \{m_v(1+L) + t_v\}\|p_{v-1} - q^\star\| + t_v L(1+L)^2\|p_v - q^\star\|.\end{aligned} \tag{3.6}$$

Using (3.6) and (3.5), we get

$$
\begin{aligned}
\|p_v - q^\star\|^2 \leq & (1-m_v)^2\|p_{v-1} - q^\star\|^2 + 2m_vL[\{m_v(1+L)+t_n\}\|p_{v-1} - q^\star\| \\
& + t_vL(1+L)^2\|p_v - q^\star\|]\|p_v - q^\star\| + 2m_v\langle M_{i(v)}^{k(v)}p_v - q^\star, j(p_v - q^\star)\rangle \\
= & (1-m_v)^2\|p_{v-1} - q^\star\|^2 + 2m_vL\{m_v(1+L)+t_v\} \\
& \|p_{v-1} - q^\star\|\|p_v - q^\star\| \\
& + 2m_vt_vL^2(1+L)^2\|p_v - q\|^2 \\
& + 2m_v\langle M_{i(v)}^{k(v)}p_v - q^\star, j(p_v - q^\star)\rangle.
\end{aligned} \tag{3.7}
$$

It is well known that

$$
\|p_{v-1} - q^\star\|\|p_v - q^\star\| \leq \frac{1}{2}(\|p_{v-1} - q^\star\|^2 + \|p_v - q^\star\|^2). \tag{3.8}
$$

By (3.7) and (3.8), we obtain

$$
\begin{aligned}
\|p_v - q^\star\|^2 \leq & (1-m_v)^2\|p_{v-1} - q^\star\|^2 + 2m_vL\{m_v(1+L)+t_v\} \\
& \times\frac{1}{2}(\|p_{v-1} - q^\star\|^2 + \|p_v - q^\star\|^2) \\
& + 2m_vt_vL^2(1+L)^2\|p_v - q^\star\|^2 \\
& + 2m_v\langle M_{i(v)}^{k(v)}p_v - q^\star, j(p_v - q^\star)\rangle \\
\leq & [(1-m_v)^2 + m_vL\{m_v(1+L)+t_v\}]\|p_{v-1} - q^\star\|^2 \\
& + [2m_vt_vL^2(1+L)^2 + m_vL\{m_v(1+L)+t_v\}]\|p_v - q^\star\|^2 \\
& + 2m_v\langle M_{i(v)}^{k(v)}p_v - q^\star, j(p_v - q^\star)\rangle.
\end{aligned} \tag{3.9}
$$

Since T_i $(i \in I)$ are TAP mappings, from (3.9) we obtain

$$
\begin{aligned}
\|p_v - q^\star\|^2 \leq & [(1-m_v)^2 + m_vL\{m_v(1+L)+t_v\}]\|p_{v-1} - q^\star\|^2 \\
& + [2m_vt_vL^2(1+L)^2 + m_vL\{m_v(1+L)+t_n\}]\|p_v - q^\star\|^2 \\
& + 2m_v(\|p_v - q^\star\|^2 + \mu_v\phi(\|p_v - q^\star\|) + \xi_v) \\
= & [(1-m_v)^2 + m_vL\{m_v(1+L)+t_v\}]\|p_{v-1} - q^\star\|^2 \\
& + [2m_vt_vL^2(1+L)^2 + m_vL\{m_v(1+L)+t_v\} + 2m_v]\|p_v - q^\star\|^2 \\
& + 2m_v\mu_v\phi(\|p_v - q^\star\|) + 2m_v\xi_v.
\end{aligned} \tag{3.10}
$$

Since ϕ is a strictly increasing function, we know that $\phi(e) \leq \phi(K)$, if $e \leq K$; $\phi(e) \leq K^*e^2$, if $e \geq K$. In either case, one can have

$$
\phi(e) \leq \phi(K) + K^*e^2. \tag{3.11}
$$

From (3.10) and (3.11), we have

$$\begin{aligned}\|p_v - q^\star\|^2 \leq & [(1-m_v)^2 + m_v L\{m_v(1+L)+t_v\}]\|p_{v-1} - q^\star\|^2 \\ & + [2m_v t_v L^2(1+L)^2 + m_v L\{m_v(1+L)+t_v\} \\ & + 2m_v]\|p_v - q^\star\|^2 + 2m_v\mu_n\phi(K) + 2m_v K^*\mu_n\|p_v - q^\star\|^2 + 2m_v\xi_v \\ = & [(1-m_v)^2 + m_v L\{m_v(1+L)+t_v\}]\|p_{v-1} - q^\star\|^2 \\ & + [2m_v t_v L^2(1+L)^2 + m_v L\{m_v(1+L)+t_v\} \\ & + 2m_v + 2m_v K^*\mu_v]\|p_v - q^\star\|^2 + 2m_v\mu_v\phi(K) + 2m_v\xi_v \\ = & R_v\|p_{v-1} - q^\star\|^2 + S_v\|t_v - q^\star\|^2 + 2m_v\mu_v\phi(K) + 2m_v\xi_v,\end{aligned} \tag{3.12}$$

where

$$\begin{aligned}R_v &= (1-m_v)^2 + m_v L\{m_v(1+L)+t_v\}, \\ S_v &= 2m_v t_v L^2(1+L)^2 + m_v L\{m_v(1+L)+t_v\} + 2m_v + 2m_v K^*\mu_v.\end{aligned}$$

From (3.12), we obtain

$$\begin{aligned}\|p_v - q^\star\|^2 \leq & \frac{R_v}{1-S_v}\|p_{v-1} - q^\star\|^2 + \frac{2m_v\mu_v\phi(K)}{1-S_v} + \frac{2m_v\xi_v}{1-S_v} \\ = & \left(1 + \frac{R_v + S_v - 1}{1-S_v}\right)\|p_{v-1} - p\|^2 + \frac{2m_v\mu_v\phi(M)}{1-S_v} \\ & + \frac{2m_v\xi_v}{1-S_v}.\end{aligned} \tag{3.13}$$

Notice that

$$\begin{aligned}R_v + S_v - 1 = & m_v^2 + m_v L\{m_v(1+L)+t_v\} + 2m_v t_v L^2(1+L)^2 \\ & + m_v L\{m_v(1+L)+t_v\} + 2m_v K^*\mu_v.\end{aligned} \tag{3.14}$$

Now, set

$$V_v = R_v + S_v - 1. \tag{3.15}$$

Since $\lim\limits_{v\to\infty} m_v = 0$, by assumptions (ii)-(iv), we obtain

$$\begin{aligned}S_v = & 2m_v t_v L^2(1+L)^2 + m_v L\{m_v(1+L)+t_v\} \\ & + 2m_v + 2m_v K^*\mu_v \to 0 \text{ as } v \to \infty,\end{aligned}$$

it implies that a positive integer n_0 exists such that

$$\frac{1}{2} < 1 - S_v \leq 1, \ \forall\ v \geq n_0.$$

Therefore, (3.13) yields

$$\begin{aligned}\|p_v - q^\star\|^2 &\le (1 + 2V_v)\|p_{v-1} - q^\star\|^2 + 4m_v\mu_v\phi(K) + 4m_v\xi_v \\ &= (1 + U_v)\|p_{v-1} - q^\star\|^2 + K_v, \ \forall\ v \ge n_0,\end{aligned} \tag{3.16}$$

where

$$\begin{aligned}U_v &= 2K_v,\\ K_v &= 4m_v\mu_v\phi(K) + 4m_v\xi_v.\end{aligned}$$

By conditions (ii)–(iv), we can easily see that $\sum_{v=1}^{\infty} U_v < \infty$ and $\sum_{v=1}^{\infty} K_v < \infty$. Obviously, by (3.16), it follows that all the assumptions of Lemma 2.3 are performed. Hence, $\lim_{v\to\infty} \|p_v - q^\star\|$ exists for all $q^\star \in \Im$. □

Theorem 3.2 *Let $\mathcal{Q}$ be a Banach space and $\mathcal{W}$ be a nonempty close convex subset of $\mathcal{Q}$.Let $i \in I = [1, N]$, where $N \in \mathbb{N}$. Let $M_i : \mathcal{W} \to \mathcal{W}$ be a finite family of uniformly L_m^i–Lipschitzian TAP mappings with sequences $\{a_v^i\} \subset [0,\infty)$ and $\{b_v^i\} \subset [0,\infty)$, where $a_v^i \to 0$ and $b_v^i \to 0$ as $v \to \infty$, $H_i : \mathcal{W} \to \mathcal{W}$ be a finite family of uniformly L_h^i–Lipschitzian TAP mappings with sequences $\{f_v^i\} \subset [0,\infty)$ and $\{d_v^i\} \subset [0,\infty)$, where $f_v^i \to 0$ and $d_v^i \to 0$ as $v \to \infty$ and $G_i : \mathcal{W} \to \mathcal{W}$ be a finite family of uniformly L_g^i–Lipscitzian TAP mappings with sequences $\{\eta_v^i\} \subset [0,\infty)$ and $\{l_v^i\} \subset [0,\infty)$, where $\eta_v^i \to 0$ and $l_v^i \to 0$ as $n \to \infty$, for each $i \in I$. Let $\mu_v = \max\{a_v, f_v, \eta_v\}$, where $a_v = \max\{a_v^i : i \in I\}$, $f_v = \max\{f_v^i : i \in I\}$ and $\eta_v = \max\{\eta_v^i : i \in I\}$. Let $\xi_v = \max\{b_v, d_v, l_v\}$, where $b_v = \max\{\lambda_v^i : i \in I\}$, $d_v = \max\{d_v^i : i \in I\}$ and $l_v = \max\{l_v^i : i \in I\}$. Suppose $\Im = (\bigcap_{i=1}^N F(M_i)) \bigcap (\bigcap_{i=1}^N F(H_i)) \bigcap (\bigcap_{i=1}^N F(G_i)) \neq \emptyset$. Let $\phi(r) = \max\{\phi_i(r) : i \in I\}$, for each $r \ge 0$. Assume that there exist $K, K^* > 0$ such that $\phi(e) \le K^* e^2$ for all $e \ge K$. Let $\{m_v\}$, $\{t_v\}$ and $\{c_v\}$ be sequences in [0,1]. If the following assumptions are performed:*

(i) $\sum_{v=1}^{\infty} m_v = \infty$;

(ii) $\sum_{v=1}^{\infty} m_v^2 < \infty$;

(iii) $\sum_{n=1}^{\infty} m_v\mu_v < \infty$, $\sum_{v=1}^{\infty} m_v\xi_v < \infty$;

(iv) $\sum_{v=1}^{\infty} m_v t_v < \infty$;

(v) $m_v t_v L^2[1 + v_v(L-1)] < 1$, $\forall v \ge 1$, where $L = \max\{L_m^1, \ldots, L_m^N, L_h^1, \ldots, L_h^N, L_g^1, \ldots, L_g^N\}$.

Let $\{p_v\}$ be a sequence defined by (1.10). Then, $\{p_v\}$ converges strongly to a point in $\Im$ if and only if

$$\liminf_{n\to\infty} d(p_v, \Im) = 0, \tag{3.17}$$

where $d(p, \Im)$ stands for the distance of p to set $\Im$, that is, $d(p, \Im) = \inf_{q^\star \in \Im} d(p, q^\star)$.

Proof The necessity of condition (3.17) is trivial.

Now, we show the sufficiency of Theorem 3.2. Given $q^\star \in \Im$. By (3.16) in Lemma 2.3, we obtain

$$[d(p_v, \Im)]^2 \le (1 + U_v)[d(p_{v-1}, \Im)]^2 + K_v, \ \forall v \ge v_0. \tag{3.18}$$

Using conditions $(ii) - (iv)$, we have $\sum_{v=1}^{\infty} U_v < \infty$ and $\sum_{v=1}^{\infty} K_v < \infty$. From (3.18) and Lemma 2.3 we know that $\lim_{v\to\infty}[d(p_v, , \Im)]^2$ exists, further, $\lim_{v\to\infty} d(p_v, \Im)$ exists. From condition (3.17), we obtain

$$\lim_{n\to\infty} d(p_v, \Im) = 0. \tag{3.19}$$

Next we show that $\{p_v\}$ is a Cauchy sequence in $\mathcal{W}$. Since $\sum_{v=1}^{\infty} K_v < \infty$, then $1 + p \le e^p$ for all $p > 0$ and by (3.16) we therefore have

$$\|p_v - q^\star\|^2 \le e^{U_v}\|p_{v-1} - q^\star\|^2 + K_v, \ \ge n_0. \tag{3.20}$$

Thus, given any positive integers $v, s \ge v_0$, from (3.20) we obtain

$$\begin{aligned} \|p_{v+s} - q^\star\|^2 &\le e^{U_{v+s}}\|p_{v+s-1} - q^\star\|^2 + K_{v+s} \\ &\le e^{U_{v+s}}[e^{U_{v+s-1}}\|p_{v+s-2} - q^\star\|^2 + K_{v+s-1}] + K_{v+s} \\ &\le e^{U_{v+s}+U_{v+s-1}}\|p_{v+s-2} - q^\star\|^2 + K_{v+s-1}] + K_{v+s} \\ &\le \cdots \\ &\le e^{\sum_{i=v+1}^{v+s} U_i}\|p_v - q^\star\|^2 + e^{\sum_{i=v+2}^{v+s} U_i}\sum_{i=v+1}^{v+s} K_i \\ &\le \zeta\|p_v - q^\star\|^2 + \zeta\sum_{i=v+1}^{\infty} K_i, \end{aligned} \tag{3.21}$$

where $\zeta = e^{\sum_{v=1}^{\infty} U_v} < \infty$.

Since $\lim_{n\to\infty} d(p_v, \Im) = 0$ and $\lim_{v\to\infty} K_v < \infty$, there exists a positive integer $v_1 \ge v_0$ such that for any given $\epsilon > 0$, we have

$$[d(p_v, \Im)]^2; \frac{\epsilon^2}{8(\zeta + 1)}, \quad \sum_{i=v+1}^{\infty} K_i; \frac{\epsilon^2}{4\zeta}, \quad \forall v \ge v_1. \tag{3.22}$$

Thus, there exists $p_1 \in \Im$ such that

$$\|p_v - p_1\|^2 < \frac{\epsilon^2}{8(\zeta+1)}, \quad \forall\, v \geq v_1. \tag{3.23}$$

It follows that for any $v \geq v_1$ and for all $s \geq 1$ we have

$$\begin{aligned}\|p_{v+s} - p_v\|^2 &\leq 2(\|p_{v+s} - p_1\|^2 + \|p_v - p_1\|^2)\\ &\leq 2(1+\zeta)\|p_v - p_1\|^2 + 2\zeta \sum_{i=v+1}^{\infty} \varphi_i\\ &< 2 \cdot \frac{\epsilon^2}{4(\zeta+1)}(1+\zeta) + 2\zeta \cdot \frac{\epsilon^2}{4\zeta}\\ &= \epsilon^2,\end{aligned}$$

i.e.,

$$\|p_{v+s} - p_v\| < \epsilon.$$

It follows that the sequence $\{p_v\}$ is Cauchy in $\mathcal{W}$. Since $\mathcal{W}$ is complete, one can assume that $p_v \to q_1^* \in \mathcal{W}$.

Next, we show that $q_1^* \in \Im$. Proving by contradiction, we assume that q_1^* is not in $\Im = (\bigcap_{i=1}^N F(M_i)) \bigcap (\bigcap_{i=1}^N F(H_i)) \bigcap (\bigcap_{i=1}^N F(G_i)) \neq \emptyset$. Since $\Im$ is a closed subset of $\mathcal{Q}$, we have that $d(q_1^*, \Im) > 0$. Thus, for all $q^* \in \Im$, we have

$$\|q_1^* - q^\star\| \leq \|q_1^* - p_v\| + \|p_v - q^\star\|, \tag{3.24}$$

which implies that

$$d(q_1^*, \Im) \leq \|p_v - q_1^*\| + d(p_v, \Im), \tag{3.25}$$

thus, we have $d(q_1^*, \Im) = 0$ as $v \to \infty$, which contradicts to $d(q_1^*, \Im) > 0$. Hence, $q_1^* \in \Im$. This completes the proof. □

The following results are derived from Theorem 3.2

Corollary 3.3 *Let $\mathcal{Q}$ be a Banach space and $\mathcal{W}$ be a nonempty close convex subset of $\mathcal{Q}$. Let $i \in I = [1, N]$, where $N \in \mathbb{N}$. Let $M_i : \mathcal{W} \to \mathcal{W}$ be a finite family of uniformly L_m^i–Lipschitzian TAP mappings with sequences $\{a_v^i\} \subset [0, \infty)$ and $\{b_v^i\} \subset [0, \infty)$, where $a_v^i \to 0$ and $b_v^i \to 0$ as $v \to \infty$. Let $\mu_v = \max\{a_v^i : i \in I\}$, $\xi_v = \max\{b_v^i : i \in I\}$. Suppose $\Im = \bigcap_{i=1}^N F(M_i) \neq \emptyset$. Let $\phi(r) = \max\{\phi_i(r) : i \in I\}$, for each $r \geq 0$. Assume that there exist $K, K^* > 0$ such that $\phi(e) \leq K^* e^2$ for all $e \geq K$. Let $\{p_v\}$ be the sequence defined by*

$$\begin{cases} p_0 \in \mathcal{W}, \\ p_v = (1 - m_v)p_{v-1} + m_v M_{i(v)}^{k(v)} w_v, \\ w_v = (1 - t_v)p_{v-1} + t_v M_{i(v)}^{k(v)} z_v, \\ z_v = (1 - c_v)p_v + c_v M_{i(v)}^{k(v)} p_v, \end{cases} \quad v \in \mathbb{N}, \tag{3.26}$$

where $\{m_v\}, \{t_v\}$ and $\{c_v\}$ are real sequences in [0,1] and $v = (k-1)N + i$, $i = v(i) \in I$, $k = k(v) \geq 1$ is some positive integers and $k(v) \to \infty$ as $v \to \infty$. If the following assumptions are performed:

(i) $\sum_{v=1}^{\infty} m_v = \infty$;

(ii) $\sum_{v=1}^{\infty} m_v^2 < \infty$;

(iii) $\sum_{v=1}^{\infty} m_v \mu_v < \infty$, $\sum_{v=1}^{\infty} m_v \xi_v < \infty$;

(iv) $\sum_{v=1}^{\infty} m_v t_v < \infty$;

(v) $m_v t_v L^2[1 + c_v(L-1)] < 1$, $\forall v \geq 1$, where $L = \max\{L_m^1, ..., L_m^N\}$.

Then, $\{p_v\}$ converges strongly to a point in $\Im$ if and only if

$$\liminf_{n \to \infty} d(p_v, \Im) = 0, \tag{3.27}$$

where $d(p, \Im)$ denotes the distance of p to set $\Im$, that is, $d(p, \Im) = \inf_{q^\star \in \Im} d(p, q^\star)$.

Proof Put $M_i = H_i = G_i$ in Theorem 3.2, then we obtain the desired result. $\square$

Corollary 3.4 *Let $\mathcal{Q}$ be a Banach space and $\mathcal{W}$ be a nonempty close convex subset of $\mathcal{Q}$. Let $i \in I = [1, N]$, where $N \in \mathbb{N}$. Let $M_i : \mathcal{W} \to \mathcal{W}$ be a finite family of uniformly L_m^i–Lipschitzian TAP mappings with sequences $\{a_v^i\} \subset [0, \infty)$ and $\{b_v^i\} \subset [0, \infty)$, where $a_v^i \to 0$ and $b_v^i \to 0$ as $v \to \infty$. Let $\mu_v = \max\{a_v^i : i \in I\}$, $\xi_v = \max\{b_v^i : i \in I\}$. Suppose $\Im = \bigcap_{i=1}^{N} F(M_i) \neq \emptyset$. Let $\phi(r) = \max\{\phi_i(r) : i \in I\}$, for each $r \geq 0$. Assume that there exist $K, K^* > 0$ such that $\phi(e) \leq K^* e^2$ for all $e \geq K$. Let $\{p_v\}$ be the sequence defined by*

$$\begin{cases} p_0 \in \mathcal{W}, \\ p_n = (1 - m_v)p_{v-1} + m_v M_{i(v)}^{k(v)} w_v, \\ w_n = (1 - t_v)p_{v-1} + t_v M_{i(v)}^{k(v)} p_v, \end{cases} \quad v \in \mathbb{N}, \tag{3.28}$$

where $\{m_v\}$, $\{t_v\}$ are real sequences in [0,1] and $n = (k-1)N + i$, $i = v(i) \in I$, $k = k(v) \geq 1$ is some positive integers and $k(v) \to \infty$ as $n \to \infty$. If the following assumptions are performed:

(i) $\sum_{v=1}^{\infty} m_v = \infty$;

(ii) $\sum_{v=1}^{\infty} m_v^2 < \infty$;

(iii) $\sum_{v=1}^{\infty} m_v \mu_v < \infty$, $\sum_{v=1}^{\infty} m_v \xi_v < \infty$;

(iv) $\sum_{v=1}^{\infty} m_v t_v < \infty$;

(v) $m_v t_v L^2 < 1$, $\forall v \geq 1$, where $L = \max\{L_m^1, \ldots, L_m^N\}$.

Then, $\{p_v\}$ converges strongly to a point in $\Im$ if and only if

$$\liminf_{v \to \infty} d(p_v, \Im) = 0, \tag{3.29}$$

where $d(p, \Im)$ denotes the distance of p to set $\Im$, that is, $d(p, \Im) = \inf_{q^\star \in \Im} d(p, q^\star)$.

Proof Put $v_n = 0$ in Corollary 3.3, then we obtain the required result. □

Corollary 3.5 *Let $\mathcal{Q}$ be a Banach space and $\mathcal{W}$ be a nonempty close convex subset of $\mathcal{Q}$. Let $i \in I = [1, N]$, where $N \in \mathbb{N}$. Let $M_i : \mathcal{W} \to \mathcal{W}$ be a finite family of uniformly L_m^i–Lipschitzian TAP mappings with sequences $\{a_v^i\} \subset [0, \infty)$ and $\{b_v^i\} \subset [0, \infty)$, where $a_v^i \to 0$ and $b_v^i \to 0$ as $v \to \infty$. Let $\mu_v = \max\{a_v^i : i \in I\}$, $\xi_v = \max\{b_v^i : i \in I\}$. Suppose $\Im = \bigcap_{i=1}^N F(M_i) \neq \emptyset$. Let $\phi(r) = \max\{\phi_i(r) : i \in I\}$, for each $r \geq 0$. Assume that there exist $K, K^* > 0$ such that $\phi(e) \leq K^* e^2$ for all $e \geq K$. Let $\{p_v\}$ be the sequence defined by*

$$\begin{cases} p_0 \in \mathcal{W}, \\ p_n = (1 - m_v) p_{v-1} + m_v M_{i(v)}^{k(v)} p_{v-1}, \end{cases} \quad v \in \mathbb{N}, \tag{3.30}$$

where $\{m_v\}$ is a real sequence in [0,1] and $v = (k-1)N + i$, $i = v(i) \in I$, $k = k(v) \geq 1$ is some positive integers and $k(v) \to \infty$ as $n \to \infty$. If the following assumptions are performed:

(i) $\sum_{v=1}^{\infty} m_v = \infty$;

(ii) $\sum_{v=1}^{\infty} m_v^2 < \infty$;

(iii) $\sum_{n=1}^{\infty} m_v \mu_v < \infty$, $\sum_{v=1}^{\infty} m_v \xi_v < \infty$.

Then, $\{p_v\}$ converges strongly to a point in $\Im$ if and only if

$$\liminf_{n \to \infty} d(p_v, \Im) = 0, \tag{3.31}$$

where $d(p, \Im)$ denotes the distance of p to set $\Im$, that is, $d(p, \Im) = \inf_{q^\star \in \Im} d(p, q^\star)$.

Proof Put $t_v = 0$ in Corollary 3.4, then we obtain the required result. □

Corollary 3.6 *Let $\mathcal{Q}$ be a Banach space and $\mathcal{W}$ be a nonempty close convex subset of $\mathcal{Q}$. Let $i \in I = [1, N]$, where $N \in \mathbb{N}$. Let $M_i : \mathcal{W} \to \mathcal{W}$ be a finite family of uniformly L^i_m–Lipschitzian TAP mappings with sequences $\{a^i_v\} \subset [0, \infty)$ and $\{b^i_v\} \subset [0, \infty)$, where $a^i_v \to 0$ and $b^i_v \to 0$ as $v \to \infty$, $H_i : \mathcal{W} \to \mathcal{W}$ be a finite family of uniformly L^i_h–Lipschitzian TAP mappings with sequences $\{f^i_v\} \subset [0, \infty)$ and $\{d^i_v\} \subset [0, \infty)$, where $f^i_v \to 0$ and $d^i_v \to 0$ as $v \to \infty$. Let $\mu_v = \max\{a_v, f_v\}$, where $a_v = \max\{a^i_v : i \in I\}$ and $f_v = \max\{f^i_v : i \in I\}$. Let $\xi_v = \max\{b_v, d_v\}$, where $b_v = \max\{b^i_n : i \in I\}$ and $d_v = \max\{d^i_v : i \in I\}$. Suppose $\Im = (\bigcap_{i=1}^N F(M_i)) \bigcap(\bigcap_{i=1}^N F(H_i)) \neq \emptyset$. Let $\phi(r) = \max\{\phi_i(r) : i \in I\}$, for each $r \geq 0$. Assume that there exist $K, K^* > 0$ such that $\phi(e) \leq K^* e^2$ for all $e \geq K$. Let $\{p_v\}$ be the sequence defined by*

$$\begin{cases} p_0 \in \mathcal{W}, \\ p_v = (1 - m_v)p_{n-1} + m_v M^{k(v)}_{i(v)} w_v, \quad n \in \mathbb{N}, \\ w_n = (1 - t_v)p_{v-1} + t_v H^{k(v)}_{i(v)} p_v, \end{cases} \tag{3.32}$$

where $\{m_v\}$ and $\{t_v\}$ are real sequences in [0,1] and $v = (k-1)N + i$, $i = v(i) \in I$, $k = k(v) \geq 1$ is some positive integers and $k(v) \to \infty$ as $v \to \infty$. If the following assumptions are performed:

(i) $\sum_{v=1}^{\infty} m_v = \infty$;

(ii) $\sum_{v=1}^{\infty} m_v^2 < \infty$;

(iii) $\sum_{v=1}^{\infty} m_v \mu_v < \infty$, $\sum_{v=1}^{\infty} m_v \xi_v < \infty$;

(iv) $\sum_{v=1}^{\infty} m_v t_v < \infty$;

(v) $m_v t_v L^2 < 1$, $\forall v \geq 1$, where $L = \max\{L^1_m, \ldots, L^N_m, L^1_h, \ldots, L^N_h\}$.

Then, $\{p_v\}$ converges strongly to a point in $\Im$ if and only if

$$\liminf_{v \to \infty} d(p_v, \Im) = 0, \tag{3.33}$$

where $d(p, \Im)$ denotes the distance of p to set $\Im$, i.e., $d(p, \Im) = \inf_{q^\star \in \Im} d(p, q^\star)$.

Proof Put $c_v = 0$ in Theorem (), then we obtain the needed result. □

These are just but a few of the numerous results that can be obtain from Theorem 3.2

4 Conclusion

In this manuscript, we have studied the class of TAP mappings which is known to be superclass of the classes of nonexpansive mappings and classes pseudocontractive mappings which have been studied in [25–27, 29, 33, 36]. Moreover, since our new iterative method contains those studied in [25–27, 29, 33, 36], it follows that our results improve, complement, generalize and extend the corresponding results in [25–27, 29, 33, 36] and many other prominent results in the current literature.

Availability of data and material The data used to support the findings of this study are included within the article.

Declarations

Conflicts of interest The authors declare no conflict of interests.

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