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Bound state solutions of the generalized shifted Hulthén potential

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Abstract: In this study, we obtain an approximate solution of the Schrödinger equation in arbitrary dimensions for the generalized shifted Hulthén potential model within the framework of the Nikiforov–Uvarov method. The bound state energy eigenvalues were computed, and the corresponding eigenfunction was also obtained. It is found that the numerical eigenvalues were in good agreement for all three approximations scheme used. Special cases were considered when the potential parameters were altered, resulting in Hulthén potential and Woods–Saxon Potential, respectively. Their energy eigenvalues expressions agreed with the already existing literature. A straightforward extension to the s-wave case for Hulthén potential and Woods–Saxon potential cases is also presented.

Keywords: Schrodinger equation; Shifted, Hulthén potential; Generalized shifted Hulthén potential; Nikiforov–Uvarov (NU) method

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1. Introduction

In the last two decades, theoretical physicists have made unprecedented progress in the study of the behavior of different quantum mechanical systems [1]. Apparently, this progress has been made possible by obtaining exact or approximate solutions of the nonrelativistic and relativistic wave equations for different physical potentials of interest. The exact or approximate solutions of these equations with central potentials play a crucial role in quantum mechanics [2–5].

The analytical solution of Schrödinger equation with $\ell = 0$ and $\ell \neq 0$ for some physical potentials has been addressed by many researchers. Some of these exponential-type potentials include, Manning–Rosen potential [6–9], Eckart potential [10–12], Poschl Teller Like Potential [13, 14], a hyperbolic potential [15–17], generalized Morse potential [18], the Morse potential [19] and screen Coulomb potential [20].

The radial Schrödinger equation for these potentials can be solved exactly for $\ell = 0$ (s-wave) but cannot be solved for these potentials for $\ell \neq 0$. To obtain the solution for $\ell \neq 0$, we employ the Pekeris-type approximation scheme to deal with the centrifugal term or solve numerically [21].The most widely used approximation was introduced by Pekeris [22], and another form was suggested by Greene and Aldrich [23] and Qiang et al. [24].

Several methods have been employed to obtain the solutions of the nonrelativistic wave equations with a chosen potential model. These includes the Nikiforov–Uvarov method (NU) [25–28], Qiang–Dong proper quantization rule [29], Factorization Method [30–32], supersymmetry quantum mechanics (SUSYQM) [33–35], Asymptotic Iteration Method (AIM) [36, 37], algebraic approach [38], etc.

The Hulthén potential [39, 40] plays a vital role in atomic and molecular physics [41]. It has also been used to explain the electronic properties of some alkali halides. [42] More so, it resembles the Coulomb interaction in structure. The Hulthén potential is one of the important short-range potentials (i.e., large b and small l [43]) in

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physics. The potential has been used in nuclear and particle physics, atomic physics, solid-state physics and its bound state and scattering properties have been investigated by employing numerous techniques. General wave functions of this potential have been used in solid-state and atomic physics problems. It should be noted that Hulthén potential is a special case of Eckart potential.

The shifted Hulthén potential has not received great attention from researchers. The Dirac equation with this potential has been investigated [44]. Recently, Ikot et al. [45] used SUSYQM approach to solve the Dirac equation with this potential in the presence of the Yukawa-like tensor (YLT) and generalized tensor (GLT) interactions. Ikot et al. [46] obtained the approximate analytical solutions of the Dirac equation for this potential within the framework of spin and pseudospin symmetry limits for arbitrary spin–orbit quantum number k using the supersymmetry quantum mechanics.

The Shifted Hulthén potential is given as [44–46];

$$V(r) = \frac{(V_0 + 1/2b^2)\mathrm{e}^{-r/b}}{1 - \mathrm{e}^{-r/b}} + \frac{V_1\mathrm{e}^{-2r/b}}{(1 - \mathrm{e}^{-r/b})^2} \tag{1}$$

which differs from the special potential Hulthén potentials [44] by the second term on the right-hand side. In the potential relation, b, is the range of the potential, V_0 and V_1 represents the depth of the potential well [44]. If $V_1 = 0$, $(V_0 + 1/2b^2) = -(V_0 + 1/2b^2)$, Eq. (1) reduces to the special Hulthén potential [46], and this is also different from the usual Hulthén potential with a new term $-1/2b^2$.

Motivated by the success in obtaining analytical solution of the Dirac Equation (Relativistic Quantum Mechanics) with this potential Eq. (1) using the standard method by Jian et al. [44] and Supersymmetric Quantum Mechanics (SUSYQM) method by Ikot et al. [45, 46], we attempt to modify the shifted Hulthén potential by introducing a deformation parameter (q), and solve for this potential using a different method called the Nikiforov–Uvarov method (N–U) [47]. The essence of introducing the deformation parameter is to have a wider range of applications. The proposed potential (generalized shifted Hulthén potential) is given by;

$$V_q(r) = \frac{(V_0 + 1/2b^2)\mathrm{e}^{-r/b}}{1 - q\mathrm{e}^{-r/b}} + \frac{V_1\mathrm{e}^{-2r/b}}{(1 - q\mathrm{e}^{-r/b})^2} \tag{2}$$

In the potential relation, b is the range of the potential and V_0 represents the depth of the potential well. Bear in mind the outcomes of the Kratzer potential in the ad hoc inverse square term for small distances [46].

The short-range generalized shifted Hulthén potential will be solved within the framework of the Pekeris-type approximations suggested by [24] to solve the Schrödinger equation (nonrelativistic Quantum Mechanics) for any arbitrary ℓ-state. These approximations are [32, 48, 49]:

$$\frac{1}{r^2} \approx \frac{\mathrm{e}^{-r/b}}{b^2(1 - q\mathrm{e}^{-r/b})^2} \tag{3}$$

Equation (3) is the commonly used approximation [24];

$$\frac{1}{r^2} \approx \frac{1}{b^2}\left[\frac{\mathrm{e}^{(1-r)/b}}{(1 - q\mathrm{e}^{-r/b})} + \frac{\mathrm{e}^{-2r/b}}{(1 - q\mathrm{e}^{-r/b})^2}\right] \tag{4}$$

and the one suggested by [24];

$$\frac{1}{r^2} \approx \frac{1}{b^2}\left[\frac{1}{12} + \frac{\mathrm{e}^{-r/b}}{(1 - q\mathrm{e}^{-r/b})^2}\right] \tag{5}$$

Equations (4) and (5) are more general than Eqs. (3), (4) and (5) which give a better approximation to the centrifugal term when b is small [24].

In view of the above, the research reported in the present paper was also motivated by the fact that the nonrelativistic treatment of the shifted Hulthén potential has not been reported in the available literature.

This paper is organized as follows. In Sect. 2, the review of the Nikiforov–Uvarov Method is presented. In Sect. 3, this method is applied to solve the radial Schrödinger equation with the generalized shifted Hulthén potential. In Sect. 4, numerical calculations are given, the results are compared for the three approximations understudy, and we discuss the results. In Sect. 5, special cases are presented, comments are made, and we give a brief concluding remark in Sect. 6.

2. Review of Nikiforov–Uvarov method

The Nikiforov–Uvarov (NU) method is based on solving the hypergeometric-type second-order differential equations by means of the special orthogonal functions. The main equation which is closely associated with the method is given in the following form [43]

$$\psi''(s) + \frac{\tilde{\tau}(s)}{\sigma(s)}\psi'(s) + \frac{\tilde{\sigma}(s)}{\sigma^2(s)}\psi(s) = 0 \tag{6}$$

where $\sigma(s)$ and $\tilde{\sigma}(s)$ are polynomials at most second degree, $\tilde{\tau}(s)$ is a first-degree polynomial, and $\psi(s)$ is a function of the hypergeometric type.

The exact solution of Eq. (6) can be obtained by using the transformation

$$\psi(s) = \phi(s)y(s) \tag{7}$$

This transformation reduces Eq. (6) into a hypergeometric-type equation of the form

$$\sigma(s)y''(s) + \tau(s)y'(s) + \lambda y(s) = 0 \tag{8}$$

The function $\phi(s)$ can be defined as the logarithm derivative

$$\frac{\phi'(s)}{\phi(s)} = \frac{\pi(s)}{\sigma(s)} \tag{9}$$

$$\text{where } \pi(s) = \frac{1}{2}[\tau(s) - \tilde{\tau}(s)] \tag{10}$$

with $\pi(s)$ being at most a first-degree polynomial. The second $\psi(s)$ being $y_n(n)$ in Eq. (7), is the hypergeometric function with its polynomial solution given by Rodrigues relation

$$y^{(n)}(s) = \frac{B_n}{\rho(s)}\frac{\mathrm{d}^n}{\mathrm{d}s^n}[\sigma^n \rho(s)] \tag{11}$$

Here B_n is the normalization constant, and $\rho(s)$ is the weight function which must satisfy the condition

$$(\sigma(s)\rho(s))' = \sigma(s)\tau(s) \tag{12}$$

$$\tau(s) = \tilde{\tau}(s) + 2\pi(s) \tag{13}$$

It should be noted that the derivative of $\tau(s)$ with respect to s should be negative. The eigenfunctions and eigenvalues can be obtained using the definition of the following function $\pi(s)$ and parameter λ, respectively:

$$\pi(s) = \frac{\sigma'(s) - \tilde{\tau}(s)}{2} \pm \sqrt{\left(\frac{\sigma'(s) - \tilde{\tau}(s)}{2}\right)^2 - \tilde{\sigma}(s) + k\sigma(s)} \tag{14}$$

$$\text{where } k = \lambda - \pi'(s) \tag{15}$$

The value of k can be obtained by setting the discriminant of the square root in Eq. (9) equal to zero. As such, the new eigenvalue equation can be given as

$$\lambda_n = -n\tau'(s) - \frac{n(n-1)}{2}\sigma''(s), \quad n = 0, 1, 2, \ldots \tag{16}$$

3. Bound State Solution

The radial Schrodinger equation in D dimension can be given as [50]:

$$\left[\frac{\mathrm{d}^2 R_{nl}}{\mathrm{d}r^2} - \frac{2\mu V_q(r)}{\hbar^2} - \frac{(D+2\ell-1)(D+2\ell-3)}{4r^2} + \frac{2\mu E_{nl}}{\hbar^2}\right] R_{nl}(r) = 0 \tag{17}$$

where μ is the reduced mass, E_{nl} is the energy spectrum, $\hbar$ is the reduced Planck's constant, and n and l are the radial and orbital angular momentum quantum numbers, respectively (or vibration–rotation quantum number in quantum chemistry). Substituting Eq. (2) into Eq. (17) gives:

$$\left[\frac{\mathrm{d}^2 R_{nl}}{\mathrm{d}r^2} - \frac{2\mu}{\hbar^2}\left(\frac{(V_0 + 1/2b^2)\mathrm{e}^{\frac{r}{b}}}{1 - q\mathrm{e}^{\frac{r}{b}}} + \frac{V_1 \mathrm{e}^{\frac{2r}{b}}}{(1 - q\mathrm{e}^{\frac{r}{b}})^2}\right) - \frac{(D+2\ell-1)(D+2\ell-3)}{4r^2} + \frac{2\mu E_{nl}}{\hbar^2}\right] R_{nl}(r) = 0 \tag{18}$$

Simplifying further Eq. 18 becomes:

$$\left[\frac{\mathrm{d}^2 R_{nl}}{\mathrm{d}r^2} - \frac{2\mu}{\hbar^2}\left(\frac{(V_0 + 1/2b^2)\mathrm{e}^{\frac{r}{b}}}{1 - q\mathrm{e}^{\frac{r}{b}}} + \frac{V_1 \mathrm{e}^{\frac{2r}{b}}}{(1 - q\mathrm{e}^{\frac{r}{b}})^2}\right) - \frac{(D+2\ell-1)(D+2\ell-3)}{4}\left(\frac{\alpha^2 \mathrm{e}^{-\alpha r}}{(1 - q\mathrm{e}^{-\alpha r})^2}\right) + \frac{2\mu E_{nl}}{\hbar^2}\right] R_{nl}(r) = 0 \tag{19}$$

Employing the Pekeris type (approximation 1) (Eq. 3) and $\alpha = \frac{1}{b}$, Eq. (19) becomes:

$$\frac{\mathrm{d}^2 R_{n\ell}(r)}{\mathrm{d}r^2} + \frac{1}{(1 - q\mathrm{e}^{-2\alpha r})^2}\left[\frac{2\mu E_{nl}}{\hbar^2}(1 - q\mathrm{e}^{-2\alpha r})^2 - \frac{2\mu}{\hbar^2}\left(\left(V_0 + \frac{\alpha^2}{2}\right)\mathrm{e}^{-\alpha r}(1 - q\mathrm{e}^{-\alpha r})\right) - \frac{2\mu}{\hbar^2}(V_1 \mathrm{e}^{2\alpha r}) - \frac{(D+2\ell-1)(D+2\ell-3)}{4}(\alpha^2 \mathrm{e}^{-\alpha r})\right] R_{n\ell}(r) \tag{20}$$

Equation (20) can be simplified further by introducing the following dimensionless abbreviations

$$\left\{\begin{array}{c} -\varepsilon_n = \frac{2\mu E_{nl}}{\hbar^2 \alpha^2} \\ \beta = \frac{2\mu V_1}{\hbar^2 \alpha^2} \\ \chi = -\frac{2\mu\left(V_0 + \frac{\alpha^2}{2}\right)}{\hbar^2 \alpha^2} \\ \eta = \frac{(D+2\ell-1)(D+2\ell-3)}{4} \end{array}\right\} \tag{21}$$

And using the transformation $s = \mathrm{e}^{-\alpha r}$ so as to enable us apply the NU method as a solution of the hypergeometric type

$$\frac{d^2 R_{n\ell}(r)}{dr^2} = \alpha^2 s^2 \frac{d^2 R_{n\ell}(s)}{ds^2} + \alpha^2 s \frac{dR_{n\,\ell}(s)}{ds} \tag{22}$$

$$\frac{\mathrm{d}^2 R_{n\ell}(s)}{\mathrm{d}s^2} + \frac{1 - qs}{s(1 - qs)}\frac{\mathrm{d}R_{n\ell}(s)}{\mathrm{d}s} + \frac{1}{s^2(1 - qs)^2}\left[-s^2(\varepsilon_n q^2 + \chi q - \beta) + s(2\varepsilon_n q + \chi - \eta) - \varepsilon_n\right] R_{n\ell}(s) = 0 \tag{23}$$

Comparing Eqs. (6) and (23), we have the following parameters

$$\left\{\begin{array}{l} \tilde{\tau}(s) = 1 - qs \\ \sigma(s) = s(1 - qs) \\ \tilde{\sigma}(s) = -s^2(\varepsilon_n q^2 + \chi q - \beta) + s(2\varepsilon_n q + \chi - \eta) - \varepsilon_n \end{array}\right\} \tag{24}$$

Substituting these polynomials into Eq. (14), we get $\pi(s)$ to be

$$\pi(s) = -\frac{qs}{2} \pm \sqrt{(a-k)s^2 + (b+k)s + c} \tag{25}$$

where

$$\left\{ \begin{array}{c} a = \frac{q^2}{4} + \varepsilon_n q^2 + \chi q - \beta \\ b = -(2\varepsilon_n q + \chi - \eta) \\ c = \varepsilon_n \end{array} \right\} \tag{26}$$

To find the constant k, the discriminant of the expression under the square root of Eq. (25) must be equal to zero. As such, we have that

$$k_{\pm} = -(\eta - \chi) \pm 2\sqrt{\varepsilon_n \left(\frac{q^2}{4} + \beta + \eta q\right)} \tag{27}$$

Substituting Eq. (27) into Eq. (25) yields

$$\pi(s) = -\frac{qs}{2} \pm \left[\left(q\sqrt{\varepsilon_n} + \sqrt{\left(\frac{q^2}{4} + \beta + \eta q\right)}\right)s - \sqrt{\varepsilon_n}\right] \tag{28}$$

From the knowledge of NU method, we choose the expression $\pi(s)_-$ which the function $\tau(s)$ has a negative derivative. This is given by

$$k_- = -(\eta - \chi) - 2\sqrt{\varepsilon_n \left(\frac{q^2}{4} + \beta + \eta q\right)} \tag{29}$$

with $\tau(s)$ being obtained as

$$\tau(s) = 1 - 2qs - 2\left[\left(\sqrt{\left(\frac{q^2}{4} + \beta + \eta q\right)} + q\sqrt{\varepsilon_n}\right)s - \sqrt{\varepsilon_n}\right] \tag{30}$$

Referring to Eq. (15), we define the constant λ as

$$\lambda = -\frac{q}{2} - \left(q\sqrt{\varepsilon_n} + \sqrt{\left(\frac{q^2}{4} + \beta + \eta q\right)}\right) - (\eta - \chi) - 2\sqrt{\varepsilon_n \left(\frac{q^2}{4} + \beta + \eta q\right)} \tag{31}$$

Taking the derivative of $\tau(s)$ with respect to s in Eq. (30), we get:

$$\tau'(s) = -2\left(q + \left(\sqrt{\left(\frac{q^2}{4} + \beta + \eta q\right)} + q\sqrt{\varepsilon_n}\right)\right) < 0 \tag{32}$$

From Eq. (24), taking the derivative of $\sigma(s)$ with respect to s, we get:

$$\sigma''(s) = -2q \tag{33}$$

Substituting Eqs. (28) and (29) into Eq. (16) and carrying out simple algebra, we get: λ_n. Setting $\lambda_n = \lambda$ and carrying out some algebraic manipulations, we have:

$$\varepsilon_n = \frac{1}{4}\left[\frac{\left(n + \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{\beta}{q^2} + \frac{\eta}{q}}\right)^2 - \frac{\beta}{q^2} - \frac{\chi}{q}}{\left(n + \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{\beta}{q^2} + \frac{\eta}{q}}\right)}\right]^2 \tag{34}$$

$$E_{n\ell}^{\text{Approx.1}} = -\frac{\hbar^2\alpha^2}{8\mu}\left[\frac{\left(n + \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2\mu V_1}{\hbar^2\alpha^2 q^2} + \frac{(D+2\ell-1)(D+2\ell-3)}{4q}}\right)^2 - \frac{2\mu V_1}{\hbar^2\alpha^2 q^2} + \frac{2\mu\left(V_0 + \frac{\alpha^2}{2}\right)}{\hbar^2\alpha^2 q}}{\left(n + \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2\mu V_1}{\hbar^2\alpha^2 q^2} + \frac{(D+2\ell-1)(D+2\ell-3)}{4q}}\right)}\right]^2 \tag{35}$$

Substituting Eq. (21) into Eq. (34) yields the energy eigenvalue equation of the generalized shifted Hulthén potential in D dimension in the form

Again by using approximation (2) and repeating the above procedure, we can consequently obtain the energy eigenvalues as:

$$E_{n\ell}^{\text{Approx.2}} = -\frac{\hbar^2\alpha^2}{8\mu}\left[\frac{\left(n + \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2\mu V_1}{\hbar^2\alpha^2 q^2} + \frac{(D+2\ell-1)(D+2\ell-3)}{4q^2}}\right)^2 - \frac{2\mu V_1}{\hbar^2\alpha^2 q^2} + \frac{2\mu\left(V_0 + \frac{\alpha^2}{2}\right)}{\hbar^2\alpha^2 q} + \frac{(D+2\ell-1)(D+2\ell-3)}{4}\left(\frac{e^{\alpha}}{q} - \frac{1}{q^2}\right)}{\left(n + \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2\mu V_1}{\hbar^2\alpha^2 q^2} + \frac{(D+2\ell-1)(D+2\ell-3)}{4q^2}}\right)}\right]^2 \tag{36}$$

Again by using approximation (3) and repeating the above procedure, we can consequently obtain the energy eigenvalues as

$$E_{n\ell}^{\text{Approx.3}} = \frac{\hbar^2\alpha^2}{2\mu}\left(\left(\frac{(D+2\ell-1)(D+2\ell-3)C_0}{4}\right) - \frac{1}{4}\right.$$
$$\left.\left[\frac{\left(n+\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{2\mu V_1}{\hbar^2\alpha^2q^2}+\frac{(D+2\ell-1)(D+2\ell-3)}{4q}}\right)^2 - \frac{2\mu V_1}{\hbar^2\alpha^2q^2} + \frac{2\mu\left(V_0+\frac{\alpha^2}{q}\right)}{\hbar^2\alpha^2q}}{\left(n+\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{2\mu V_1}{\hbar^2\alpha^2q^2}+\frac{(D+2\ell-1)(D+2\ell-3)}{4q}}\right)}\right]^2\right) \tag{37}$$

The corresponding wave functions can be evaluated by substituting $\pi(s)\_ and\, \sigma(s)$ from Eqs. (24) and (28), respectively, into Eq. (9) and solving the first-order differential equation. This gives

$$\Phi(s) = s^{\sqrt{\varepsilon_n}}(1-qs)^{\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{\beta}{q^2}+\frac{\eta}{q}}} \tag{38}$$

The weight function $\rho(s)$ from Eq. (12) can be obtained as

$$\rho(s) = s^{2\sqrt{\varepsilon_n}}(1-qs)^{2\sqrt{\frac{1}{4}+\frac{\beta}{q^2}+\frac{\eta}{q}}} \tag{39}$$

From the Rodrigues relation of Eq. (11), we obtain

$$y_n(s) \equiv N_{n,l}P_n^{\left(2\sqrt{\varepsilon_n},2\sqrt{\frac{1}{4}+\frac{\beta}{q^2}+\frac{\eta}{q}}\right)}(1-2qs) \tag{40}$$

where $P_n^{(\theta,\vartheta)}$ is the Jacobi Polynomial.

Substituting $\Phi(s)$ and $y_n(s)$ from Eqs. (32) and (34), respectively, into Eq. (3), we obtain

$$R_{nl}(s) = N_{n,l}s^{\sqrt{\varepsilon_n}}(1-qs)^{\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{\beta}{q^2}+\frac{\eta}{q}}}P_n^{\left(2\sqrt{\varepsilon_n},2\sqrt{\frac{1}{4}+\frac{\beta}{q^2}+\frac{\eta}{q}}\right)}(1-2qs) \tag{41}$$

4. Results and Discussion

To show the accuracy of our results, we obtained the eigenvalues (in units of fm^{-1}.) numerically (Tables 1, 2, 3) for arbitrary quantum numbers n and l with the potential parameter $\alpha = 0.025, 0.050$ and $0.075\ \text{fm}^{-1}$ in 3D. In Table 1, we present the numerical results for generalized shifted Hulthén potential in natural units for $q = 1$ [absence of deformation (generalized shifted Hulthén)], potential strength, $V_0 = 5\,\text{fm}^{-1}$ and $V_1 = 2\,\text{fm}^{-1}$ and $\alpha = {}^1/_b = 0.025, 0.050$ and $0.075\,\text{fm}^{-1}$.The domain in which the screening parameter $\alpha \to 0\ \text{fm}^{-1}$ is called the low screening regime. In this regime, the generalized shifted Hulthén potential model becomes the constant. For a fixed value of angular momentum quantum l, the energy spectrum increases as the principal quantum number n increases for the strong potential coupling strengths,$V_0 = 5$ and $V_1 = 2\,\text{fm}^{-1}$ as seen in Table 1. For a fixed value of angular momentum quantum l, the energy spectrum increases as the principal quantum number n increases for a small screening parameter (i.e., low screening regime) "α". An increase in angular momentum quantum l leads to an increase in the energy spectrum as the principal quantum number n increases for a varying screening parameter α and for a strong potential coupling strength, (V_0 and V_1). For a weak potential coupling strength, (V_0 and V_1), solutions are ignored due to the presence of imaginary terms and the

Table 1 The bound state energy levels (in units of fm^{-1}) of the generalized shifted Hulthén potential for various values of n, l and for $\hbar = \mu = 1$, $q = 1$, $V_0 = 5$, $V_1 = 2$ and ${}^1/_b = 0.025, 0.050$ and 0.075

n	l	${}^1/_b$	Approx. 1	Approx. 2	Approx. 3
0	1	0.025	− 3.117514261	− 3.11753389	− 3.117462178
		0.050	− 3.110700106	− 3.110857879	− 3.110491773
		0.075	− 3.104584691	− 3.10511967	− 3.104115941
1	1	0.025	− 3.103082968	− 3.10310231	− 3.103030885
		0.050	− 3.084157322	− 3.084310638	− 3.083948989
		0.075	− 3.06815173	− 3.068664704	− 3.06768298
2	1	0.025	− 3.089786861	− 3.089805928	− 3.089734778
		0.050	− 3.061877318	− 3.062026488	− 3.061668985
		0.075	− 3.040735741	− 3.041228935	− 3.040266991
0	2	0.025	− 3.117139376	− 3.117198239	− 3.116983126
		0.050	− 3.109268049	− 3.109740695	− 3.108643049
		0.075	− 3.101523451	− 3.103123335	− 3.100117201
1	2	0.025	− 3.102737046	− 3.102795052	− 3.102580796
		0.050	− 3.082946589	− 3.083405908	− 3.082321589
		0.075	− 3.065803052	− 3.067337454	− 3.064396802
2	2	0.025	− 3.089468696	− 3.089525875	− 3.089312446
		0.050	− 3.060870455	− 3.061317376	− 3.060245455
		0.075	− 3.039019378	− 3.040494886	− 3.037613128
0	3	0.025	− 3.116578656	− 3.116696314	− 3.116266156
		0.050	− 3.107144798	− 3.108088059	− 3.105894798
		0.075	− 3.09705253	− 3.100237354	− 3.09424003
1	3	0.025	− 3.102219691	− 3.102335639	− 3.101907191
		0.050	− 3.08115308	− 3.082069834	− 3.07990308
		0.075	− 3.062385156	− 3.065440549	− 3.059572656
2	3	0.025	− 3.088992914	− 3.089107212	− 3.088680414
		0.050	− 3.059380745	− 3.060272836	− 3.058130745
		0.075	− 3.036536508	− 3.039475458	− 3.033724008
0	4	0.025	− 3.115834011	− 3.116029961	− 3.115313178
		0.050	− 3.104359635	− 3.105927274	− 3.102276302
		0.075	− 3.091311559	− 3.096587012	− 3.086624059
1	4	0.025	− 3.10153273	− 3.101725838	− 3.101011897
		0.050	− 3.078803444	− 3.080327222	− 3.076720111
		0.075	− 3.058019492	− 3.063082576	− 3.053331992
2	4	0.025	− 3.088361249	− 3.088551611	− 3.087840416
		0.050	− 3.057432485	− 3.058915442	− 3.055349152
		0.075	− 3.033393131	− 3.038265041	− 3.028705631
3	4	0.025	− 3.076273284	− 3.076461004	− 3.075752451
		0.050	− 3.039925015	− 3.04136992	− 3.037841682
		0.075	− 3.016497519	− 3.021196724	− 3.011810019

Table 2 The bound state energy levels (in units of fm^{-1}) of the generalized shifted Hulthén potential for various values of , and for $\hbar = \mu = 1, q = 2$, $V_0 = 5, V_1 = 2$ and $\alpha = 0.025$

n	l	Approx. 1	Approx. 2	Approx. 3
0	1	− 3.07804408	− 3.07900825	− 3.077991997
1	1	− 2.991656822	− 2.992562348	− 2.991604739
2	1	− 2.911896391	− 2.912748234	− 2.911844308
3	1	− 2.838174895	− 2.83897748	− 2.838122812
0	2	− 3.075800058	− 3.07868871	− 3.075643808
1	2	− 2.989586095	− 2.992299144	− 2.989429845
2	2	− 2.909983401	− 2.912535695	− 2.909827151
3	2	− 2.836405828	− 2.838810616	− 2.836249578
0	3	− 3.072445134	− 3.078210908	− 3.072132634
1	3	− 2.986490142	− 2.991905695	− 2.986177642
2	3	− 2.907123184	− 2.912218111	− 2.906810684
3	3	− 2.833760719	− 2.838561429	− 2.833448219
0	4	− 3.067992528	− 3.077576635	− 3.067471695
1	4	− 2.982381021	− 2.991383619	− 2.981860188
2	4	− 2.903326769	− 2.911796942	− 2.902805936
3	4	− 2.830249678	− 2.838231239	− 2.829728845

Table 3 The bound state energy levels (in units of fm^{-1}) of the generalized shifted Hulthén potential for various values of , and for $\hbar = \mu = 1, q = -2$, $V_0 = 5, V_1 = 2$ and $\alpha = 0.025$

n	l	Approx. 1	Approx. 2	Approx. 3
0	1	− 3.020072465	− 3.017282372	− 3.020020382
1	1	− 2.817314959	− 2.814748239	− 2.817262876
2	1	− 2.629306878	− 2.626941798	− 2.629254795
3	1	− 2.454721581	− 2.452538949	− 2.454669498
0	2	− 3.025345822	− 3.016966052	− 3.025189572
1	2	− 2.822200951	− 2.814492116	− 2.822044701
2	2	− 2.633840768	− 2.626737579	− 2.633684518
3	2	− 2.458934669	− 2.452379472	− 2.458778419
0	3	− 3.033281168	− 3.016493064	− 3.032968668
1	3	− 2.829553036	− 2.814109248	− 2.829240536
2	3	− 2.640662724	− 2.626432419	− 2.640350224
3	3	− 2.46527366	− 2.452141296	− 2.46496116
0	4	− 3.043909145	− 3.01586518	− 3.043388312
1	4	− 2.839399168	− 2.813601206	− 2.838878335
2	4	− 2.649798308	− 2.626027709	− 2.649277475
3	4	− 2.473761971	− 2.45182566	− 2.473241138

energy spectrum is not complex but real. Beyond this, we can observe from Table 1, the energy eigenvalue is strongly bounded and an increase in rotational quantum number l makes energy become more attractive (i.e., the energy becomes more negative) with increasing α.

Interestingly, the above observation is the same in the presence of the deformation parameter in the system as shown in Tables 2 and 3 for $q = 2$ and $q = -2$ except for the fact that the presence of the deformation parameter makes the energy become more attractive. However, the energy is more attractive when the deformation parameter is less than 0 $(q < 0)$. The analytical expressions for the total energy levels of this system are found to be general in the sense that it is obtained in arbitrary dimensions and the presence of the deformation parameter provides an avenue to arrive at special cases, e.g., when $q \rightarrow -q$, we arrive at the Woods–Saxon potential, etc.

In this study, three approximation schemes were employed. To show that Eqs. (3–5) are good approximation scheme, we compared $\frac{1}{r^2}$ and the approximation scheme with $\alpha = 0.025$ in Fig. 1 for $q = 1$. In Fig. 2, the variation of shifted Hulthén potential, special Hulthén potential and Hulthén potential with r for $V_0 = 5$, $V_1 = 2$ and $\alpha = 0.025$ was plotted. This was done in order to enable us to show the behavior of the shifted Hulthén potential. It can be easily observed that the Hulthén and special Hulthén behave in the same manner. Figures 3, 4, 5 and 6 show the behavior of the wave function in the presence and absence of the deformation parameter. Finally, we point out that these exact results obtained for this newly proposed form of the potential (2) may have some interesting applications in the study of different quantum mechanical systems, atomic and molecular physics.

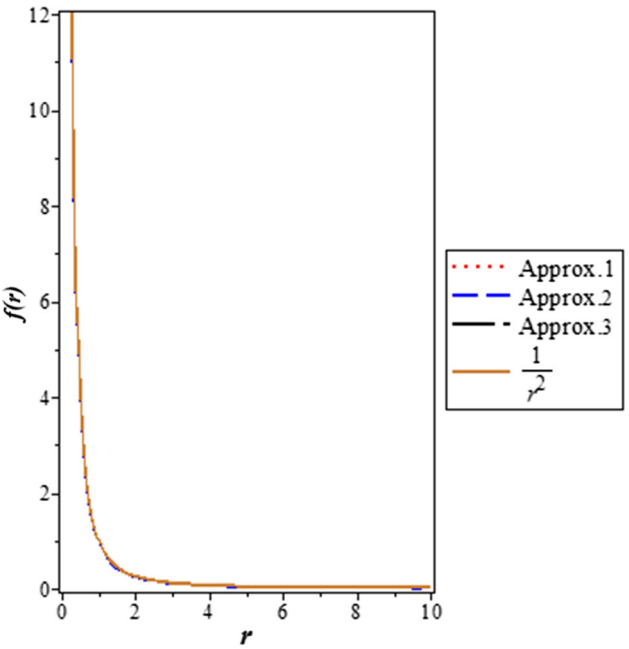

Fig. 1 Comparison between $1/r^2$ and the approximation scheme as functions of r for $\alpha = 0.025$

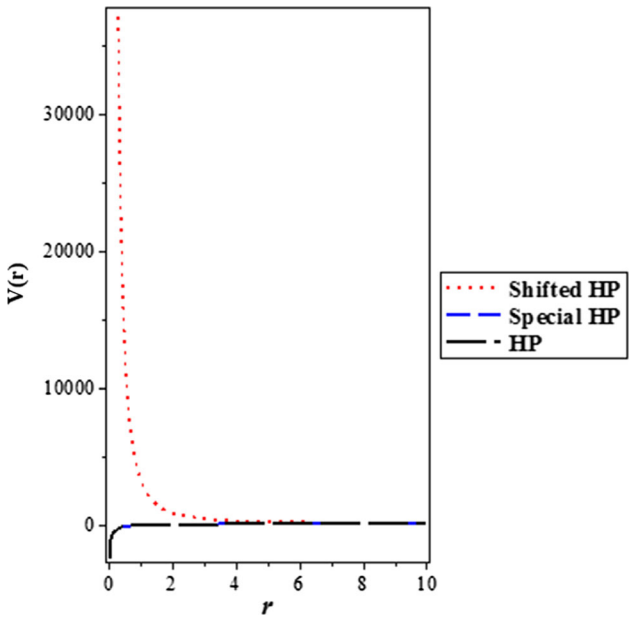

Fig. 2 Variation of shifted Hulthén potential, Special Hulthén potential and Hulthén potential with r for $V_0 = 5, V_1 = 2$ and $\alpha = 0.025$

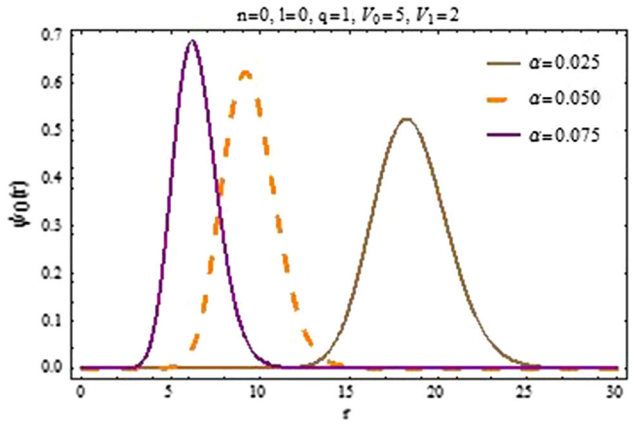

Fig. 3 Wave functions against radial distance for different values of the screening parameter (α). We chose $n = \ell = 0, q = 1, V_0 = 5$ and $V_1 = 2$

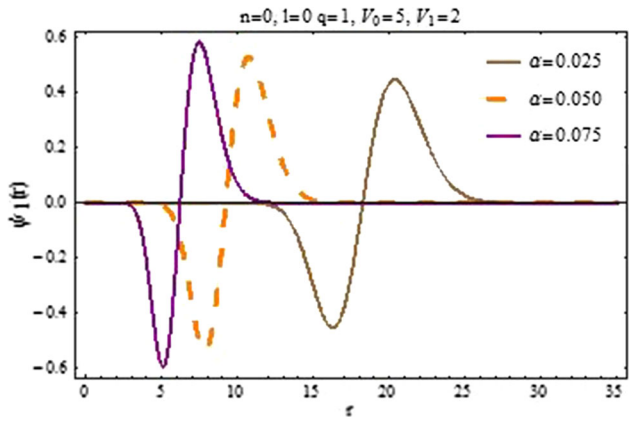

Fig. 4 Wave functions against radial distance for different values of the screening parameter (α). $n = 1,\ \ell = 0, q = 1, V_0 = 5$ and $V_1 = 2$

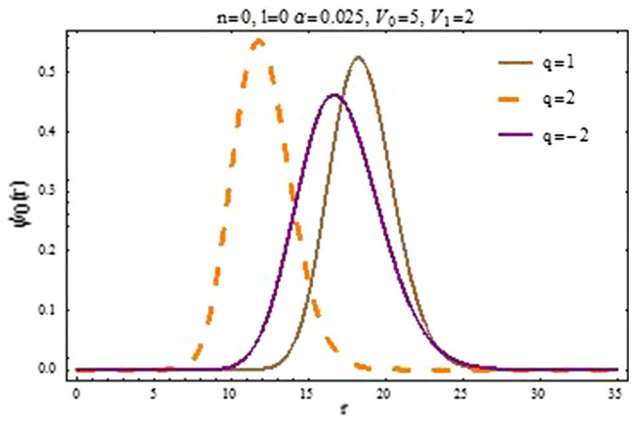

Fig. 5 Wave functions against radial distance for different values of the deformation parameter (q). We chose $n = \ell = 0$, $\alpha = 0.025, V_0 = 5$ and $V_1 = 2$

5. Special case

In this section, we make some adjustments of constants in Eq. (2) and Eqs. (35), (36) and (37) to have the following cases:

First, we study the s-wave case $(\ell = 0)$ for $D = 3$ and $q = 1$. The solutions of energy eigenvalues Eqs. (35), (36) and (37) reduce to the following equation

$$E_{n\ell}^{\text{Approx.1}} = E_{n\ell}^{\text{Approx.2}} = E_{n\ell}^{\text{Approx.3}} = -\frac{\hbar^2\alpha^2}{8\mu}\left[\frac{\left(n+\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{2\mu V_1}{\hbar^2\alpha^2}}\right)^2 - \frac{2\mu V_1}{\hbar^2\alpha^2} + \frac{2\mu\left(V_0+\frac{\alpha^2}{2}\right)}{\hbar^2\alpha^2}}{\left(n+\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{2\mu V_1}{\hbar^2\alpha^2}}\right)}\right]^2 \tag{42}$$

5.1. Hulthén potential

If we set $V_1 = 0, V^1 = \frac{1}{2b^2} = 0, = \frac{1}{b}, q = 1$ and $V_0 = -V_0$ in Eq. (2), we obtain the Hulthén potential as follows;

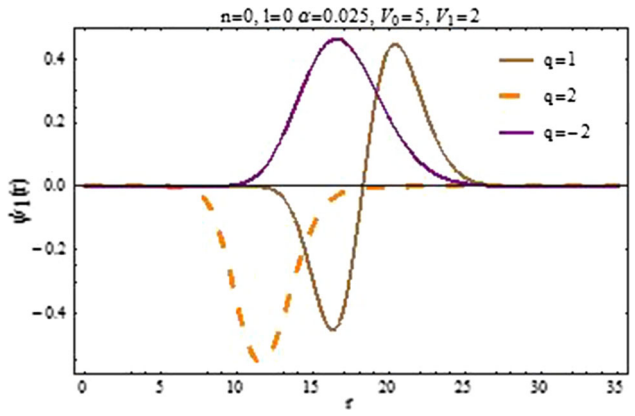

Fig. 6 Wave functions against radial distance for different values of the deformation parameter (q). We chose $n = 1,\ \ell = 0,\ \alpha = 0.025$, $V_0 = 5$ and $V_1 = 2$

$$V(r) = \frac{-V_0 e^{-\alpha r}}{1 - e^{-\alpha r}} \tag{43}$$

Its energy eigenvalue equation can be deduced from Eqs. (34), (36) and (37) as

$$E_{n\ell}^{\text{Approx.1}} = -\frac{\hbar^2\alpha^2}{8\mu}\left[\frac{\left(n+\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{(D+2\ell-1)(D+2\ell-3)}{4}}\right)^2 - \frac{2\mu V_0}{\hbar^2\alpha^2}}{\left(n+\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{(D+2\ell-1)(D+2\ell-3)}{4}}\right)}\right]^2 \tag{44}$$

$$E_{n\ell}^{\text{Approx.2}} = -\frac{\hbar^2\alpha^2}{8\mu}\left[\frac{\left(n+\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{(D+2\ell-1)(D+2\ell-3)}{4}}\right)^2 - \frac{2\mu V_0}{\hbar^2\alpha^2} + \frac{(D+2\ell-1)(D+2\ell-3)}{4}(e^{\alpha}-1)}{\left(n+\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{(D+2\ell-1)(D+2\ell-3)}{4}}\right)}\right]^2 \tag{45}$$

and

$$E_{n\ell}^{\text{Approx.3}} = \frac{\hbar^2\alpha^2}{2\mu}\left(\left(\frac{(D+2\ell-1)(D+2\ell-3)C_0}{4}\right) - \frac{1}{4}\left[\frac{\left(n+\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{(D+2\ell-1)(D+2\ell-3)}{4}}\right)^2 - \frac{2\mu V_0}{\hbar^2\alpha^2}}{\left(n+\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{(D+2\ell-1)(D+2\ell-3)}{4}}\right)}\right]^2\right) \tag{46}$$

In 3D, Eqs. (44), (45) and (46) reduce to:

$$E_{n\ell}^{\text{Approx.1}} = -\frac{\hbar^2\alpha^2}{8\mu}\left[\frac{\left(n+\frac{1}{2}+\sqrt{\frac{1}{4}+\ell(\ell+1)}\right)^2 - \frac{2\mu V_0}{\hbar^2\alpha^2}}{\left(n+\frac{1}{2}+\sqrt{\frac{1}{4}+\ell(\ell+1)}\right)}\right]^2 \tag{47}$$

$$E_{n\ell}^{\text{Approx.2}} = -\frac{\hbar^2\alpha^2}{8\mu}\left[\frac{\left(n+\frac{1}{2}+\sqrt{\frac{1}{4}+\ell(\ell+1)}\right)^2 - \frac{2\mu V_0}{\hbar^2\alpha^2} + \ell(\ell+1)(e^{\alpha}-1)}{\left(n+\frac{1}{2}+\sqrt{\frac{1}{4}+\ell(\ell+1)}\right)}\right]^2 \tag{48}$$

and

$$E_{n\ell}^{\text{Approx.3}} = \frac{\hbar^2\alpha^2}{2\mu}\left(\left(\ell(\ell+1)C_0\right) - \frac{1}{4}\left[\frac{\left(n+\frac{1}{2}+\sqrt{\frac{1}{4}+\ell(\ell+1)}\right)^2 - \frac{2\mu V_0}{\hbar^2\alpha^2}}{\left(n+\frac{1}{2}+\sqrt{\frac{1}{4}+\ell(\ell+1)}\right)}\right]^2\right) \tag{49}$$

Equation (44) is identical with the energy eigenvalue equation given in Eq. (30) of Ref. [51]. More so, if we set $D = 3$, we arrive at the energy eigenvalue equation for the Hulthén potential in 3D

Equation (47) is identical with the energy eigenvalues formula given in Eq. (31) of Ref. [51], Eq. (32) of Ref. [52], Eq. (24) of Ref. [53] and Eq. (28) of Ref. [54] and Eq. (36) of Ref. [55]. Equation (49) is identical with the energy eigenvalues formula (34) of [55]

Furthermore, for s-wave ($\ell = 0$) states, Eqs. (47), (48) and (49) reduce to

$$E_{n\ell}^{\text{Approx.1}} = E_{n\ell}^{\text{Approx.2}} = E_{n\ell}^{\text{Approx.3}} = -\frac{\hbar^2\alpha^2}{8\mu}\left[\frac{(n+1)^2 - \frac{2\mu V_0}{\hbar^2\alpha^2}}{(n+1)}\right]^2 \tag{50}$$

which is identical to the ones obtained before using the factorization method [56], SUSYQM approach [57–59], NU method [54, 55, 60] and AIM Eq. (39) of Ref [36]..

5.2. Woods–Saxon potential

If $V_1 = 0, V^1 = \frac{1}{2b^2} = 0, = \frac{1}{b}$, and $q = -1$ in Eq. (2), we can obtain the Woods–Saxon potential of the form:

$$V(r) = \frac{-V_0 e^{-\alpha r}}{1 + e^{-\alpha r}} \tag{51}$$

Its energy eigenvalue equation can be deduced from Eqs. (34), (36) and (37) as

$$E_{n\ell}^{\text{Approx.1}} = -\frac{\hbar^2\alpha^2}{8\mu}\left[\frac{\left(n+\frac{1}{2}+\sqrt{\frac{1}{4}-\frac{(D+2\ell-1)(D+2\ell-3)}{4}}\right)^2-\frac{2\mu V_0}{\hbar^2\alpha^2}}{\left(n+\frac{1}{2}+\sqrt{\frac{1}{4}-\frac{(D+2\ell-1)(D+2\ell-3)}{4}}\right)}\right]^2 \tag{52}$$

Again by using approximation (2) and repeating the above procedure, we can consequently obtain the energy eigenvalues as:

$$E_{n\ell}^{\text{Approx.2}} = -\frac{\hbar^2\alpha^2}{8\mu}\left[\frac{\left(n+\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{(D+2\ell-1)(D+2\ell-3)}{4}}\right)^2-\frac{2\mu V_0}{\hbar^2\alpha^2}-\frac{(D+2\ell-1)(D+2\ell-3)}{4}(e^{\alpha}+1)}{\left(n+\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{(D+2\ell-1)(D+2\ell-3)}{4}}\right)}\right]^2 \tag{53}$$

Again by using approximation (3) and repeating the above procedure, we can consequently obtain the energy eigenvalues as

$$E_{n\ell}^{\text{Approx.3}} = \frac{\hbar^2\alpha^2}{2\mu}\left(\left(\frac{(D+2\ell-1)(D+2\ell-3)C_0}{4}\right)-\frac{1}{4}\left[\frac{\left(n+\frac{1}{2}+\sqrt{\frac{1}{4}-\frac{(D+2\ell-1)(D+2\ell-3)}{4}}\right)^2-\frac{2\mu V_0}{\hbar^2\alpha^2}}{\left(n+\frac{1}{2}+\sqrt{\frac{1}{4}-\frac{(D+2\ell-1)(D+2\ell-3)}{4}}\right)}\right]^2\right) \tag{54}$$

Equations (52), (53) and (54) are the energy equation for Woods–Saxon potential in D Dimensions with different approximation scheme. If $D = 3$, Eqs. (52), (53) and (54) reduce to energy equation for Woods–Saxon potential in 3D. More so, Eq. (52) is in agreement with Eq. (30) of ref [61] and Eq. (64) of [62, 63]

6. Conclusion

In this work, we have studied the bound state solutions of the Schrodinger equation with generalized shifted Hulthén potential in D dimensions using NU method. We used three different approximation schemes to deal with the centrifugal term; we obtain the energy eigenvalues and the corresponding eigenfunctions and also discussed some special cases of the potential. We have calculated numerical energy eigenvalues and presented plots for various values of the potential parameters. It is found out that the numerical values were in good agreement. The results are in excellent agreement with literature. Finally, our results can find many applications in quantum mechanical systems, atomic and molecular physics.

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