A NOVEL APPLICATION OF IMAN TRANSFORM FOR SOLVING LINEAR VOLTERRA INTEGRAL EQUATIONS

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ABSTRACT
In this paper, we state and prove the convolution theorem for Iman transform. Also, we state the Iman transform of some elementary functions. The convolution theorem for Iman transform enables us to solve faltung type Volterra integral equations. The result was used to solve some linear Volterra integral equations of the second kind. The given applications showed that the exact solutions have been obtained using very less computational work and spending a very little time.

Keywords: Linear Volterra integral equation; Iman transform; Convolution theorem; Inverse Iman transform.

1.0 INTRODUCTION

Historically, the concept of an integral transform originated from the celebrated Fourier integral formula. The importance of integral transforms is that they provide powerful operational methods for solving initial value problems and initial-boundary value problems for linear differential and integral equations (Debnath & Bhatta, 2007). Using the corresponding convolution theorem for a given integral transform, we can solve Volterra integral equations.

Volterra examined the linear Volterra integral equation of the second kind of the form (Rahman, 2007; Linz, 1984; Kanwal, 1997; Wazwaz, 1997; Song & Kim, 2014):

$$u(x) = f(x) + \lambda \int_0^x k(x-t)u(t)dt \tag{1}$$

where $k(x-t)$ is the kernel of the integral equation, and λ is a parameter. Volterra integral equations appear in many applications in physical and biological sciences.

The Iman transform was proposed by (Iman, 2023). Several integral transforms have been proposed by different scholars (Watugula, 1993; Khan & Khan, 2008; Elzaki, 2011; Atangana & Kilicman, 2013; Srivasta et al., 2015; Zafar, 2016; Ramadan, et al., 2016; Yang, 2016). The Iman transform of a function $f(t)$ is defined as:

$$I\{f(t)\} = L(v) = \int_0^\infty f(t)e^{\frac{-t}{v}}dt \tag{2}$$

Ahmed & Elzaki (2013) used the combined Sumudu transform and Adomian decomposition method to solve nonlinear Volterra and integro-differential equations of the second kind. Mansour et al., (2023) used the SEE transform to solve linear Volterra integro-differential equations of the second kind. Aggarwal et al., (2018) applied Kamal transform to solve linear Volterra integral equations of the second kind. Homotopy analysis method was used to solve Volterra and integro-differential equations of the second kind in (Sha & Singh, (2015).

The aim of this work is to establish the exact solutions for linear Volterra integral equations of the second kind using Iman transform without large computational work. To achieve this aim, we shall state and prove the convolution theorem for the Iman transform and use the result to find the exact solutions for linear Volterra integral equations of the second kind.

This paper is organised as follows: in section 2, we present the Iman transform of some elementary functions. in section 3, we state the results of the Iman transform of the derivatives. In section 4, we define the convolution of two functions. In section 5, we state and prove the convolution theorem for Iman transform. In section 6, we define the inverse Iman transform. In section 7, we state the processes involved in using the Iman transform to solve linear Volterra integral equations. In section 8, some applications are given in order to demonstrate the effectiveness of the Iman transform for solving linear Volterra integral equations of the second kind. In section 9, we give concluding remarks to close the paper.

Table 1: Iman Transform of Some Elementary Functions

$S.N.$	$f(t)$	$I\{f(t)\} = L(v)$
1.	1	$\frac{1}{v^4}$
2.	t	$\frac{1}{v^6}$
3.	t^2	$\frac{2}{v^8}$
4.	$t^n, n \geq 0$	$\frac{n!}{v^{2n+4}}$
5.	e^{at}	$\frac{1}{v^4 - av^2}$
6.	$sinat$	$\frac{a}{v^2(v^4 + a^2)}$
7.	$cosat$	$\frac{1}{(v^4 + a^2)}$
8.	$sinhat$	$\frac{a}{v^2(v^4 - a^2)}$
9.	$coshat$	$\frac{1}{(v^4 - a^2)}$

Table 1 shows the Iman transform of the elementary functions that occur in the solutions of all linear Volterra integral equations of the second kind.

2.0 IMAN TRANSFORM OF THE DERIVATIVES OF $f(t)$

If $I\{f(t)\} = L(v)$, then

$$I\{f'(t)\} = v^2 L(v) - \frac{1}{v^2} f(0) \quad (3)$$

$$I\{f''(t)\} = v^4 L(v) - f(0) - \frac{1}{v^2} f'(0) \quad (4)$$

$$I\{f^{(n)}(t)\} = v^n L(v) - \sum_{k=0}^{n-1} \frac{1}{v^{4-2n+2k}} f^{(k)}(0) \quad (5)$$

2.1 Convolution of Two Functions

Convolution of two functions $f(t)$ and $g(t)$ is denoted by $f(t) * g(t)$ and it is defined by

$$f(t) * g(t) = f * g = \int_0^t f(x) g(t-x) dx \quad (6)$$

$$f(t) * g(t) = f * g = \int_0^t g(x) f(t-x) dx \quad (7)$$

2.2 Convolution Theorem for Iman Transform

If $I(f) = L(v)$ and $I(g) = M(v)$, then
$I(f * g) = v^2 I(f) I(g) = v^2 L(v) M(v)$

Proof

Let

$$I(f) = \frac{1}{v^2}\int_0^\infty e^{-v^2 x} f(x) dx \text{ and } I(g) = \frac{1}{v^2}\int_0^\infty e^{-v^2 p} g(p) dp$$

$$I(f)I(g) = \frac{1}{v^4}\int_0^\infty e^{-v^2 x} f(x) dx \int_0^\infty e^{-v^2 p} g(p) dp$$

Let us put $t = x + p,\ p = t - x$, we get

$$I(f)I(g) = \frac{1}{v^4}\int_0^\infty e^{-v^2 x} f(x) \int_p^\infty e^{-v^2 (t-x)} g(t-x) dx dt$$

$$= \frac{1}{v^4}\int_0^\infty e^{-v^2 x} f(x) \int_p^\infty e^{-v^2 t} e^{v^2 x} g(t-x) dx dt$$

$$= \frac{1}{v^4}\int_0^\infty e^{-v^2 x} e^{v^2 x} f(x) \int_p^\infty e^{-v^2 t} g(t-x) dx dt$$

$$= \frac{1}{v^4}\int_0^\infty e^{-v^2 t} \int_0^t f(x) g(t-x) dx dt$$

$$= \frac{1}{v^4}\int_0^\infty e^{-v^2 t} (f * g) dt$$

$$= \frac{1}{v^2} I(f * g)$$

$$\therefore I(f)I(g) = \frac{1}{v^2} I(f * g)$$

$$\Longrightarrow I(f * g) = v^2 I(f)I(g)$$

$$= v^2 L(v)M(v) \qquad (8)$$

2.3 Inverse Iman Transform

If $I\{f(t)\} = L(v)$, then $f(t)$ is called the inverse Iman transform of $L(v)$ and mathematically it is defined as $f(t) = I^{-1}\{L(v)\}$.

2.4 Iman Transform for Linear Volterra Integral Equations

In this work, we will assume that the kernel $k(x,t)$ of (1) is a difference kernel that can be expressed by the difference $(x - t)$. The linear Volterra equation (1) can thus be expressed as

$$u(x) = f(x) + \lambda \int_0^x k(x-t)u(t)dt \qquad (9)$$

Apply the Iman transform to both sides of (9), we have

$$I\{u(x)\} = I\{f(x)\} + \lambda I\left\{\int_0^x k(x-t)u(t)dt\right\} \qquad (10)$$

Using the convolution theorem of Iman transform, we have

$$I\{u(x)\} = I\{f(x)\} + \lambda I\{k(x)\}I\{u(x)\} \qquad (11)$$

Operating inverse Iman transform on both sides of (11), we have

$$I\{u(x)\} = I\{f(x)\} + \lambda I^{-1}\{k(x)\}I\{u(x)\} \qquad (12)$$

which is the required solution of (9).

3.0 APPLICATIONS

In this section, some applications are given in order to demonstrate the effectiveness of Iman transform for solving linear Volterra integral transform of the second kind.

Example 1: Consider the linear Volterra integral equation with $\lambda = -1$

$$u(x) = x - \int_0^x (x - t)u(t)dt \qquad (13)$$

Applying the Iman transform to both sides of (13), we have

$$I\{u(x)\} = \frac{1}{v^6} - I\left\{\int_0^x (x - t)u(t)dt\right\} \qquad (14)$$

Using convolution theorem of Iman transform on (14), we have

$$I\{u(x)\} = \frac{1}{v^6} - v^2\left(\frac{1}{v^6}\right) I\{u(x)\}$$

Which simplifies to

$$I\{u(x)\} = \frac{1}{v^2(v^4+1)} \tag{15}$$

Operating inverse Iman transform on both sides of (15), we have

$$u(x) = I^{-1}\left\{\frac{1}{v^2(v^4+1)}\right\}$$
$$= sinx \tag{16}$$

which is the required exact solution of (13).

Example 2: Consider the linear Volterra integral equation with $\lambda = -1$

$$u(x) = cosx + sinx - \int_0^x u(t)dt \tag{17}$$

Applying the Iman transform to both sides of (17), we have

$$I\{u(x)\} = \frac{1}{v^4+1} + \frac{1}{v^2(v^4+1)} - I\left\{\int_0^x u(t)dt\right\} \tag{18}$$

Using convolution theorem of Iman transform on (18), we have

$$I\{u(x)\} = \frac{1}{v^4+1} + \frac{1}{v^2(v^4+1)} - v^2 I\{1\} I\{u(x)\}$$

Which simplifies to

$$I\{u(x)\} = \frac{1}{v^4+1} \tag{19}$$

Operating inverse Iman transform on both sides of (19), we have

$$u(x) = I^{-1}\left\{\frac{1}{v^4+1}\right\} = cosx \tag{20}$$

which is the required exact solution of (17).

Example 3: Consider the linear Volterra integral equation with $\lambda = 1$

$$u(x) = 1 - x + \int_0^x (x-t)u(t)dt \tag{21}$$

Applying the Iman transform to both sides of (21), we have

$$I\{u(x)\} = \frac{1}{v^4} + \frac{1}{v^6} - I\left\{\int_0^x (x-t)u(t)dt\right\} \tag{22}$$

Using convolution theorem of Iman transform on (22), we have

$$I\{u(x)\} = \frac{1}{v^4} + \frac{1}{v^6} - v^2\left(\frac{1}{v^6}\right) I\{u(x)\}$$

Which simplifies to

$$I\{u(x)\} = \frac{1}{v^4-1} - \frac{1}{v^2(v^4+1)} \tag{23}$$

Operating inverse Iman transform on both sides of (23), we have

$$u(x) = I^{-1}\left\{\frac{1}{v^4-1} - \frac{1}{v^2(v^4+1)}\right\} = coshx - sinhx = e^{-x} \tag{24}$$

which is the required exact solution of (21).

Example 4: Consider the linear Volterra integral equation with $\lambda = -1$

$$u(x) = 1 - \int_0^x (x-t)u(t)dt \tag{25}$$

Applying the Iman transform to both sides of (25), we have

$$I\{u(x)\} = \frac{1}{v^4} - I\left\{\int_0^x (x-t)u(t)dt\right\} \tag{26}$$

Using convolution theorem of Iman transform on (26), we have

$$I\{u(x)\} = \frac{1}{v^4} - v^2\left(\frac{1}{v^6}\right)I\{u(x)\}$$

Which simplifies to

$$I\{u(x)\} = \frac{1}{v^4+1} \tag{27}$$

Operating inverse Iman transform on both sides of (27), we have

$$u(x) = I^{-1}\left\{\frac{1}{v^4+1}\right\} = cosx \tag{28}$$

which is the required exact solution of (25).

Example 5: Consider the linear Volterra integral equation with $\lambda = 1$

$$u(x) = 1 - \frac{x^2}{2} + \int_0^x u(t)dt \tag{29}$$

Applying the Iman transform to both sides of (29), we have

$$I\{u(x)\} = \frac{1}{v^4} - \frac{1}{2}\left(\frac{2}{v^8}\right) + I\left\{\int_0^x u(t)dt\right\} \tag{30}$$

Using convolution theorem of Iman transform on (30), we have

$$I\{u(x)\} = \frac{1}{v^4} - \frac{1}{2}\left(\frac{2}{v^8}\right) + v^2 I\{1\}I\{u(x)\}$$

Which simplifies to

$$I\{u(x)\} = \frac{1}{v^6} + \frac{1}{v^4} \tag{31}$$

Operating inverse Iman transform on both sides of (31), we have

$$\begin{aligned} u(x) &= I^{-1}\left\{\frac{1}{v^6} + \frac{1}{v^4}\right\} \\ &= 1 + x \end{aligned} \tag{32}$$

which is the required exact solution of (29).

4.0 CONCLUSION

In this paper, we have successfully stated and proved the convolution theorem for Iman transform and used the result to solve linear Volterra integral equations of the second kind. In (Rahman, 2007), the successive approximation method, Piccard's successive approximation method, successive substitutions method, the Adomian decomposition method, and the series method were used to solve linear Volterra integral equations of the second kind and the processes involved in using the afore-mentioned methods are tedious and laborious. The given applications using our proposed method showed that the exact solutions have been obtained using very less computational work and spending a very little time. The proposed scheme can be used to solve systems of linear Volterra integral equations.

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